MAT598 - Additional Problem Set 01

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Below I have included hints/thoughts for approaching the exercises. Specific completed solutions are student-submitted.

1. See page 28 in Lee's book for the definition. There are topological manifolds that cannot be endowed with a smooth structure (although such a fact is not intuitively obvious). The E_8 manifold is one example, although several others exist.

2. Yes, as in the case $\dim M = 0$.

3. We can identy $\mathbb{RP}^2$ with the quotient space $S^2/\sim$, where $x \sim y$ if and only if $x = \pm y$.

4. Recall that every smooth atlas on M is contained in a unique maximal smooth atlas.

5.

- **a.** This should be a straightforward computation.
- **b.** This should be a straightforward computation.
- **c.** This should be a straightforward computation.

6. This is a straightforward check of the $\mathbb{R}$-algebra axioms.

7. This sounds like a job for paritistions of unity with compact support.

8.

- **a.** This is a straightforward check of $\mathbb{R}$-linearity.
- **b.** *Proof.* ($\Rightarrow$) Suppose F is a diffeomorphism. Since F^* is linear from part (a), it suffices to show that F^* is a bijection when restricted to $C^\infty(N)$. Let $f \in C^\infty(M)$, $g \in C^\infty(N)$, and define the map

$$G : C^\infty(M) \to C^\infty(N)$$
$$f \mapsto f \circ F^{-1}.$$

It is a straightforward computation to show that G is the inverse for $F^*\big|_{C^\infty(N)}$. Thus $F^*\big|_{C^\infty(N)}$.

($\Leftarrow$) Suppose $F^*(C^\infty(N)) \subseteq C^\infty(M)$. Let $x \in M$ and choose charts (U, φ) of M containing x, and charts (V, ψ) of N containing $F(x)$ so that $F(U) \subset V$. Let $\pi_i : \mathbb{R}^{\dim N} \to \mathbb{R}$ be the canonical projection onto the i^{th} coordinate. We then have that $F^*(\pi_i \circ \psi) = \pi_i \circ \psi \circ F : M \to \mathbb{R}$ is smooth by our hypothesis. Since $\pi_i \circ \psi \circ F \circ \varphi^{-1}$ is smooth for each i, $\psi \circ F \circ \varphi^{-1}$ is smooth, whence F is smooth.

□

- **c.** Define $G : \mathbb{C}^\infty(M) \to C^\infty(N)$ be $G(f) = f \circ F^{-1}$. Show that $G = (F^*)^{-1}$ (when properly restricted). Conversely