MAT502 - Additional Problem Set 08

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1. Suppose that V is a finite-dimensional real vector space. Let $\mathscr{A}$ denote the subspace of the k-fold abstract tensor product $V^* \otimes \cdots \otimes V^*$ spanned by all elements of the form $\omega^1 \otimes \cdots \otimes \omega^k$ where $\omega^i = \omega^j$ for some $i \neq j$. (Thus $\mathscr{A}$ is the trivial subspace if $k < 2$.) Let $A^k(V^*)$ denote the quotient vector space $(V^* \otimes \cdots \otimes V^*)/\mathscr{A}$.

- **a.** Show that there is a unique isomorphism $F : A^k(V^*) \to \Lambda^k(V^*)$ such that the following diagram commutes:

$$\begin{array}{ccc} V^* \otimes \cdots \otimes V^* & \xrightarrow{\cong} & T^k(V^*) \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \text{Alt}} \\ A^k(V^*) & \underset{F}{\dashrightarrow} & \Lambda^k(V^*) \end{array}$$

  (here $\pi : V^* \otimes \cdots \otimes V^* \to A^k(V^*)$ is the projection).
- **b.** Define a wedge product on $\bigoplus_k A^k(V^*)$ by $\omega \wedge \eta = \pi(\tilde{\omega} \otimes \tilde{\eta})$, where $\tilde{\omega}$, $\tilde{\eta}$ are arbitrary tensors such that $\pi(\tilde{\omega}) = \omega$ and $\pi(\tilde{\eta}) = \eta$. Show that this wedge product is well defined, and that F takes this wedge product to the Alt convention wedge product on $\Lambda(V^*)$).

2. Prove that the diagram below commutes, and use it to give a quick proof that $\text{curl} \circ \text{grad} \equiv 0$ and $\text{div} \circ \text{curl} \equiv 0$ on $\mathbb{R}^3$

$$\begin{array}{ccccccc} C^\infty(\mathbb{R}^3) & \xrightarrow{\text{grad}} & \mathfrak{X}(\mathbb{R}^3) & \xrightarrow{\text{curl}} & \mathfrak{X}(\mathbb{R}^3) & \xrightarrow{\text{div}} & C^\infty(\mathbb{R}^3) \\ \Big\downarrow{\scriptstyle \text{Id}} & & \Big\downarrow{\scriptstyle \flat} & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle *} \\ \Omega^0(\mathbb{R}^3) & \xrightarrow[d]{} & \Omega^1(\mathbb{R}^3) & \xrightarrow[d]{} & \Omega^2(\mathbb{R}^3) & \xrightarrow[d]{} & \Omega^3(\mathbb{R}^3) \end{array}$$

3. *[Spring 2015, Problem 7]* Let M be a smooth compact oriented manifold with boundary ∂M.

- **a.** Prove that there does not exist a smooth retraction $r : M \to \partial M$.
- **b.** Let B be the closed unit ball in $\mathbb{R}^n$. Use part (a) to show that every smooth map $F : B \to B$ must have a fixed point.

4. *[Fall 2015, Problem 8]*

- **a.** Let M be a compact oriented manifold without boundary. Prove that, for all $\omega \in \Omega^k(M)$ and $\eta \in \Omega^{n-k-1}(M)$,

$$\int_M d\omega \wedge \eta = (-1)^k \int_M \omega \wedge d\eta.$$

- **b.** Show that there is a smooth vector field on S^2 that vanishes at exactly one point.
- **c.** Show that there is a smooth one-form on S^2 that vanishes at exactly one point. [Hint: Use your answer to part (b) in conjunction with any Riemannian metric on S^2.] Is it possible to find a smooth, exact one-form on S^2 that vanishes at exactly one point? Why or why not?