MAT502 - Additional Problem Set 02

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1. Suppose M is a smooth manifold with or without boundary, $p \in M$, $v \in T_pM$, and $f, g \in C^\infty(M)$. Prove the following:

a. If f is a constant function, then $vf = 0$.

b. If $f(p) = g(p) = 0$, then $v(fg) = 0$.

2. Let (x, y) denote the standard coordinates on $\mathbb{R}^2$. Verify that $(\tilde{x}, \tilde{y})$ are global smooth coordinates on $\mathbb{R}^2$, where

$$\tilde{x} = x, \qquad \tilde{y} = y + x^3.$$

Let p be the point $(1, 0) \in \mathbb{R}^2$ (in standard coordinates), and show that

$$\left.\frac{\partial}{\partial x}\right|_p \neq \left.\frac{\partial}{\partial \tilde{x}}\right|_p,$$

even thought the coordinate functions $x = \tilde{x}$ are identically equal.

3. Let $M_1, \ldots, M_k$ be smooth manifolds, and for each j, let $\pi_j : M_1 \times \cdots M_k \to M_j$ be the projection onto the M_j factor. Prove that, for any point $p = (p_1, \ldots, p_k) \in M_1 \times \cdots M_k$, the map

$$\alpha : T_p(M_1 \times \cdots \times M_k) \to T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$$

defined by

$$\alpha(v) = (d(\pi_1)_p(v), \ldots, d(\pi_k)_p(v))$$

is an isomorphism. Prove that the same is true if one of the spaces M_i is a smooth manifold with boundary.

4. Let M be a smooth manifold with or without boundary and let p be a point of M. Let $C_p^\infty(M)$ denote the algebra of germs of smooth real-valued functions at p, and let $\mathscr{D}_pM$ denote the vector space of derivations of $C_p^\infty(M)$. Define a map $\Phi : \mathscr{D}_pM \to T_pM$ by $(\Phi v)f = v([f]_p)$. Show that Φ is an isomorphism.

5. True or false? A smooth bijective map of constant rank is a diffeomorphism.

6. Give an example of a non-trivial constant rank map $\varphi : \mathbb{RP}^2 \to \mathbb{RP}^2$.

7. Let M be a nonempty smooth compact manifold. Show that there is no smooth submersion $F : M \to \mathbb{R}^k$ for any $k > 0$.

8. Let $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ be the usual projectivization map and let $q = \pi\big|_{\mathbb{S}^n} : \mathbb{S}^n \to \mathbb{RP}^n$. Show that q is a smooth covering map.