MAT598 - Additional Problem Set 01

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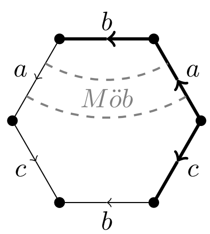
Fall 2016

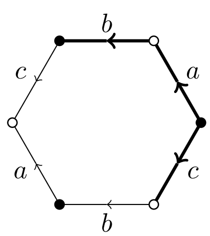
Solutions will be posted as they are submitted to me.

1.

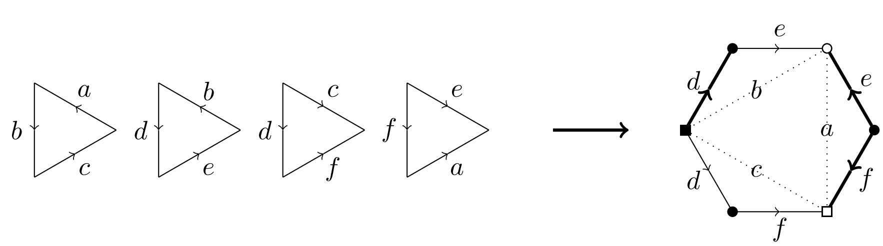
a. Let S be the surface given by the presentation. Since the a edges form a twisted pair, the surface contains a Möbius strip, and thus is nonorientable. By chasing vertices around the edges, we have that $V = 1$. Since edges are identified in pairs, we have that $E = 3$. Lastly, there is just 1 face, so $\chi(S) = V - E + F = -1$. By the classification of surfaces, we have that $S \cong 3\mathbb{P}^2 = \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$.



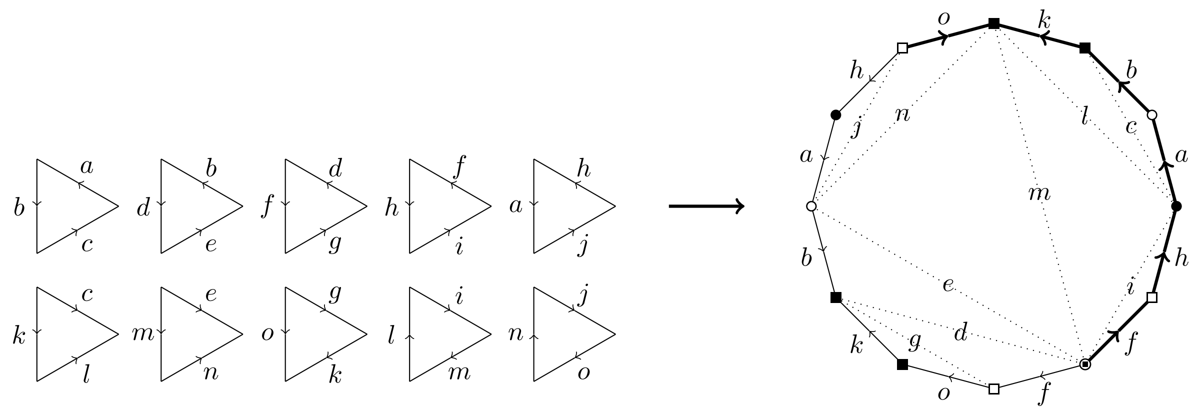
b. Let S be the surface given by the presentation. There are no twisted pairs of edges, so the surface does not contain a Möbius strip, and thus is orientable. By chasing vertices around the edges, we have that $V = 2$. Since edges are identified in pairs, we have that $E = 3$. Lastly, there is just 1 face, so $\chi(S) = V - E + F = 0$. By the classification of surfaces, we have that $S \cong \mathbb{T}^2$.



c. Although we can determine the Euler characteristic from the information given, orientability is not obvious. So, we rewrite this presentation with a single relation (this is pictorially demonstrated below) as $\langle d, e, f \mid ee^{-1}d^{-1}dff^{-1} \rangle$. From here it is clear that our presentation does not contain any twisted pairs of edges, so it is does not contain a Möbius strip, and thus is orientable. By chasing vertices around the edges, we have that $V = 4$. Since edges are identified in pairs, we have that $E = 3$. Lastly, there is just 1 face, so $\chi(S) = V - E + F = 2$. By the classification of surfaces, we have that $S \cong S^2$.



d. Although we can determine the Euler characteristic from the information given, orientability is not obvious. So, we rewrite this presentation with a single relation (this is pictorially demonstrated below) as $\langle a, b, f, h, k, o \,|\, abko^{-1}habk^{-1}o^{-1}f^{-1}fh\rangle$. From here it is clear that, since the a edges form a twisted pair, the surface contains a Möbius strip, and thus is nonorientable. By chasing vertices around the edges, we have that $V = 5$. Since edges are identified in pairs, we have that $E = 6$. Lastly, there is just 1 face, so $\chi(S) = V - E + F = 0$. By the classification of surfaces, we have that $S \cong \mathbb{P}^2 \# \mathbb{P}^2$.



2. Let V be the number of vertices, E the number of edges, and F the number of faces for our surface. Since there are m-many edges at each vertex, and there are 2 vertices on each edge, we have

$$2E = mV \quad \Rightarrow \quad V = \frac{2E}{m}.$$

Similarly, since there are n-many edges around each face, and there are 2 faces attached to each edge, we have

$$2E = nF \quad \Rightarrow \quad F = \frac{2E}{n}.$$

Since we're assuming the Euler characteristic is 2, we get

$$\begin{aligned} V - E + F &= 2 \\ \frac{2E}{m} - E + \frac{2E}{n} &= 2 \\ E\left(\frac{1}{m} - \frac{1}{2} + \frac{1}{n}\right) &= 2. \end{aligned}$$

As $E > 0$ and $n \geq 3$, we then have that

$$\begin{aligned} \frac{1}{m} - \frac{1}{2} + \frac{1}{n} &> 0 \\ \frac{1}{m} &> \frac{1}{2} - \frac{1}{n} \geq \frac{1}{2} - \frac{1}{3} = \frac{1}{6}, \end{aligned}$$

so $3 \leq m < 6$. We can consider each possible case for m. When $m = 3$, $n = 3, 4, 5$ are all possibilities. When $m = 4$, $n = 3$ is the only possibility. When $m = 5$, $n = 3$ is again the only possibility. Thus there are exactly five possibilities for Platonic solids.

3.

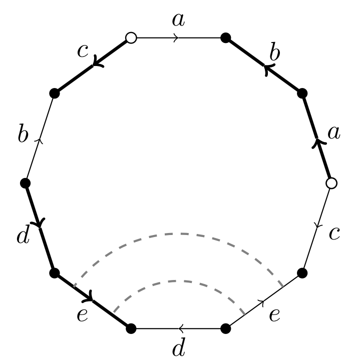
a. In order to be a surface, edges must be identified in pairs, so the number of edges $E = 4$. As well, the octagon provides a single face, hence the number of faces $F = 1$. Lastly, there must be at least one vertex, so the number of vertices $V \geq 1$. We thus have that

$$\chi(S) = V - E + F \geq 1 - 4 + 1 = -2.$$

b. *INCOMPLETE*

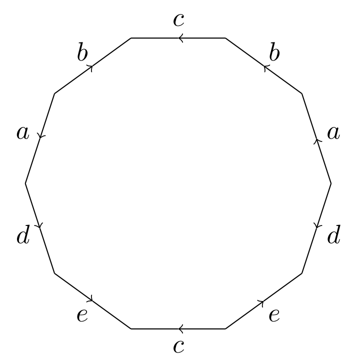
4. *INCOMPLETE*

5.



a. Yes, this is a manifold as the edges are identified in pairs.
b. No. This is nonorientable as it contains a twisted pair of edges and hence an embedded Möbius band.
c. $\chi(S) = V - E + F = 2 - 5 + 1 = -2$
d. By the classification of surfaces, a nonorientable surface is a connect sum of $2 - \chi(S)$ copies of $\mathbb{P}^2$, i.e., $S \cong \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$.

6.



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7. *INCOMPLETE*