MAT598 - Additional Problem Set 05

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1. For a covering space $p : \tilde{X} \to X$ and a subspace $A \subseteq X$, let $\tilde{A} = p^{-1}(A)$. Show that the restriction $p|_{\tilde{A}} : \tilde{A} \to A$ is a covering space.

2. Construct an uncountable number of nonisomorphic covering spaces of $S^1 \vee S^1$. Deduce that a free group on 2 generators has an uncountable number of distinct subgroups. Is this also true of the free abelian group on two generators?

3. *[May 2015]* Let F_n denote the free group on n generators. Prove that for any $n \geq 3$, F_2 contains a subgroup isomorphic to F_n. What is the index of this subgroup in F_2?

4. Let $\tilde{X}$ and $\tilde{Y}$ be simply-connected covering spaces of the path-connected, locally path-connected spaces X and Y. Show that if $X \simeq Y$, then $\tilde{X} \simeq Y$. [Hint: See *Hatcher*, Chapter 0, Exercise 10.]

5. *[August 2015]*

- **a.** Find all connected covers of T^2. Which ones are normal?
- **b.** Find all the covers $T^2 \to T^2$ and their degree.

6.

- **a.** Show that a map $f : X \to Y$ between Hausdorff spaces is a covering space if X is compact and f is a local homeomorphism, meaning that for each $x \in X$ there are open neighborhoods U of x in X and V of $f(x)$ in Y with f a homeomorphism from U to V.
- **b.** Give an example where this fails if X is noncompact.

7. Construct a simply-connected covering space for each of the following spaces:

- **a.** $S^1 \vee S^2$.
- **b.** The union of S^2 and an arc joining two distinct points of S^2.
- **c.** S^2 with two points identified.
- **d.** $\mathbb{RP}^2 \vee \mathbb{RP}^2$.
- **e.** S^2 with two arcs joining two distinct pairs of points.
- **f.** $S^1 \vee \mathbb{RP}^2$.
- **g.** $\mathbb{RP}^2$ with an arc joining two distinct points.
- **h.** $S^1 \vee T^2$, where T^2 is the torus.