MAT598 - Additional Problem Set 02

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1. *[May 2015]* Prove or disprove: in any topological space, the product of paths is associative.

2. Show that the composition of paths satisfies the following cancellation property: If $f_0 \cdot g_0 \simeq f_1 \cdot g_1$ and $g_0 \simeq g_1$, then $f_0 \simeq f_1$.

3. From the isomorphism $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$ it follows that loops in $X \times \{y_0\}$ and $\{x_0\} \times Y$ represent commuting elements of $\pi_1(X \times Y, (x_0, y_0))$. Construct an explicit homotopy demonstrating this.

4. Let H be the "southern hemisphere" in S^n: $H = \{(x_1, \ldots, x_{n+1}) \in S^n \,:\, x_{n+1} \leq 0\}$. Let p be the "south pole" $p = (0, 0, \ldots, 0, -1) \in S^n$. Show that the inclusion $(S^n, p) \to (S^n, H)$ is a homotopy equivalence of pairs.

5. If x_0 and x_1 are two points in the same path component of X, construct a bijection between the set of homotopy classes of paths from x_0 to x_1 and $\pi_1(X, x_0)$.

6. For spaces X and Y with basepoints x_0 and y_0, recall that a *basepoint-preserving map* is a map $X \to Y$ such that $x_0 \mapsto y_0$. Let $\langle X, Y \rangle$ denote the set of basepoint-preserving homotopy classes of basepoint-preserving maps $X \to Y$.

- **a.** Show that a homotopy equivalence $(Y, y_0) \simeq (Y', y_0')$ induces a bijection $\langle X, Y \rangle \approx \langle X, Y' \rangle$.
- **b.** Show that a homotopy equivalence $(X, x_0) \simeq (X', x_0)$ induces a bijection $\langle X, Y \rangle \approx \langle X', Y \rangle$.
- **c.** When X is a finite connected graph, compute $\langle X, Y \rangle$ in terms of $\pi_1(Y, y_0)$. [Use part (b) to reduce to the case that X is a wedge sum of circles.]

7. Show that if two maps $f, g : (X, x_0) \to (S^1, s_0)$ are homotopic just as maps $X \to S^1$ without regard to basepoints, then they are homotopic through basepoint-preserving maps via a homotopy $f_t : (X, x_0) \to (S^1, s_0)$.