MAT598 - Additional Problem Set 01

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1. For each of the following surface presentations, compute the Euler characteristic and determine which of our standard surfaces it represents.

- **a.** $\langle a, b, c \mid abacb^{-1}c^{-1} \rangle$
- **b.** $\langle a, b, c \mid abca^{-1}b^{-1}c^{-1} \rangle$
- **c.** $\langle a, b, c, d, e, f \mid abc, bde, c^{-1}df, e^{-1}fa \rangle$
- **d.** $\langle a, b, c, d, e, f, g, h, i, j, k, l, m, n, o \mid abc, bde, dfg, fhi, haj, c^{-1}kl, e^{-1}mn, g^{-1}ok^{-1}, i^{-1}l^{-1}m^{-1}, j^{-1}n^{-1}o^{-1} \rangle$

2. Prove that there are only five regular polyhedra (the regular tetrahedron, cube, octahedron, dodecahedron, icosahedron) by subdividing the sphere into n-gons such that exactly m-many edges meet at each vertex $(m, n \geq 3)$. Use the fact that $\chi(S^2) = 2$.

3.
- **a.** The sides of a regular octagon are identified in pairs in such a way as to obtain a compact surface, S. Prove that $\chi(S) \geq -2$.
- **b.** Prove that any surface (orientable or nonorientable) of Euler characteristic ≥ -2 can be obtained by suitably identifying in pairs the sides of a regular octagon.

4. Let S_1 be a surface that is the connected sum of $m \geq 1$ tori, and let S_2 be a surface that is the connected sum of $n \geq 1$ projective planes. Suppose two holes are cut in each of these surfaces, and the two surfaces are glued along the boundary of the holes. What is the surface obtained by this process?

5. *[May 2015]* Let S be the topological space with the polygonal presentation $\langle a, b, c, d, e \mid aba^{-1}cb^{-1}ded^{-1}ec^{-1} \rangle$.

- **a.** Is S a manifold?
- **b.** Is S orientable?
- **c.** What is the Euler characteristic $\chi(S)$?
- **d.** To which standard surface is S homeomorphic?

6. *[August 2015]* Let S be the topological space with the polygonal presentation $\langle a, b, c, d, e \mid abcb^{-1}adec^{-1}ed^{-1} \rangle$.

- **a.** Is S a manifold?
- **b.** Is S orientable?
- **c.** What is the Euler characteristic $\chi(S)$?
- **d.** To which standard surface is S homeomorphic?

7. Suppose M is a compact, connected 2-manifold that contains a subset $B \subseteq M$ that is homeomorphic to the Möbius band. Show that there is a compact 2-manifold M' such that M is homeomorphic to a connected sum $M' \# \mathbb{P}^2$. [Hint: First show there is a subset $B_0 \subseteq B$ such that $\overline{B_0}$ is homeomorphic to a Möbius band and $M - B_0$ is a compact manifold with boundary.]