GEOMETRY AND TOPOLOGY QUALIFYING EXAM: FALL 2015

Problem 1: Let S be the topological space with the polygonal presentation

$$\langle a, b, c, d, e \,|\, abcb^{-1}adec^{-1}ed^{-1} \rangle.$$

(a) Is S a manifold?

(b) Is S orientable?

(c) What is the Euler characteristic $\chi(S)$?

(d) To which standard surface is S homeomorphic?

Problem 2:

(a) Let X be a path-connected topological space, and $x, y, z \in X$. Suppose that f_0, f_1 are 2 paths from x to y and g_0, g_1 are 2 paths from y to z. Prove that if $f_0 \cdot g_0 \simeq f_1 \cdot g_1$ and $g_0 \simeq g_1$, then $f_0 \simeq f_1$ (where $\simeq$ denotes path-homotopy).

(b) Let X denote the topological space obtained by gluing the boundary circle of a Möbius band to a meridian of a torus. Find a presentation for $\pi_1(X)$.

Problem 3: Given $x_0 \in S^1$, consider the subspaces $C_1 = \{x_0\} \times S^1$ and $C_2 = S_1 \times \{x_0\}$ of the torus $T^2 = S^1 \times S^1$, and a point $p \in T^2 \setminus (C_1 \cup C_2)$.

(a) Does T^2 retract (resp. deformation retract) onto C_1 or C_2?

(b) Does T^2 retract (resp. deformation retract) onto $C_1 \cup C_2$?

(c) Does $T^2 \setminus \{p\}$ retract (resp. deformation retract) onto $C_1 \cup C_2$?

Problem 4:

(a) Find all the connected covers of T^2. Which ones are normal?

(b) Find all the covers $T^2 \longrightarrow T^2$ and their degree.

Problem 5. Let M and N be smooth nonempty manifolds and $F : M \to N$ a smooth map.

(a) Define the differential dF_p of F at a point $p \in M$.

(b) Prove that, if F is a diffeomorphism, then $\dim(M) = \dim(N)$.

(c) Suppose that M is compact and $N = \mathbb{R}^k$. Prove that, if $k > 0$, then F cannot be a smooth submersion.[1]

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[1]Hint: Consider the map $h : M \to \mathbb{R}$ given by $h(p) = |F(p)|^2$.

Problem 6.

(a) Suppose that G and H are Lie groups and $F : G \to H$ is a Lie group homomorphism. Prove that F has constant rank.

(b) Prove that $SU(n)$ is a properly embedded Lie subgroup of $U(n)$.[2] What is its dimension?

(c) Prove that $U(n)$ and $SU(n)$ are compact. Is $SL(n, \mathbb{C})$ compact?

Problem 7. In part (c), you do not need to justify your answer.

(a) Prove that every Lie group G is parallelizable (i.e., TG is trivial).

(b) Show that a smooth parallelizable manifold M is orientable.

(c) Suppose $\pi : \tilde{M} \to M$ is a smooth covering map. Under what conditions does the orientability of M imply that of $\tilde{M}$? When does the orientability of $\tilde{M}$ imply that of M?

Problem 8.

(a) Let M be a compact oriented manifold without boundary[3]. Prove that, for all $\omega \in \Omega^k(M)$ and $\eta \in \Omega^{n-k-1}(M)$,

$$\int_M d\omega \wedge \eta = (-1)^{k+1} \int_M \omega \wedge d\eta.$$

(b) Show that there is a smooth vector field on S^2 that vanishes at exactly one point.[4]

(c) Show that there is a smooth one-form on S^2 that vanishes at exactly one point.[5] Is it possible to find a smooth, exact one-form on S^2 that vanishes at exactly one point? Why or why not?

[2]Hint: Use part (a). You may assume that $U(n)$ is a Lie group.

[3]All manifolds in problems 4-8 are assumed to be without boundary.

[4]Suggestion: Use stereographic projection. For convenience, the formulas for the charts $\sigma : S^2 \setminus \{N\} \to \mathbb{R}^2$ and $\tilde{\sigma} : S^2 \setminus \{S\} \to \mathbb{R}^2$, where $N = (0,0,1)$ and $S = (0,0,-1)$, are given by

$$\sigma(x,y,z) = \frac{(x,y)}{1-z}, \quad \sigma^{-1}(u,v) = \frac{(2u, 2v, u^2+v^2-1)}{u^2+v^2+1}, \quad \tilde{\sigma}(\mathbf{x}) = -\sigma(-\mathbf{x}).$$

[5]Hint: Use your answer to (b) in conjunction with any Riemannian metric on S^2.