Recitation 12: Triple Integrals in Cylindrical/Spherical Coordinates & Integrals For Mass Calculations & Change of Variables in Multiple Integrals

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$$\text{Cartesian Coordinates} : (x, y, z)$$
$$\text{Cylindrical Coordinates} : (r, \theta, z)$$
$$\text{Spherical Coordinates} : (\rho, \phi, \theta)$$

Converting between 3-D coordinate systems:

$$\begin{array}{cc} \text{Cartesian} \to \text{Cylindrical} & \text{Cylindrical} \to \text{Cartesian} \\ r^2 = x^2 + y^2 & x = r\cos\theta \\ \tan\theta = \frac{y}{x} & y = r\sin\theta \\ z = z & z = z \end{array}$$

$$\begin{array}{cc} \text{Cartesian} \to \text{Spherical} & \text{Spherical} \to \text{Cartesian} \\ \rho^2 = x^2 + y^2 + z^2 & x = \rho\sin\phi\cos\theta \\ \phi = \text{use trig} & y = \rho\sin\phi\sin\theta \\ \theta = \text{use trig} & z = \rho\cos\phi \end{array}$$

As we saw with polar coordinates, integrating in these new coordinate systems changes our differentials slightly.

$$\begin{aligned}\iiint f\,dV &= \iiint f(x,y,z)\,dx\,dy\,dz \\ &= \iiint f(r,\theta,z)\,r\,dr\,d\theta\,dz \\ &= \iiint f(\rho,\phi,\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta\end{aligned}$$

Example. Evaluate the integral in cylindrical coordinates:

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{x^2+y^2}} \sqrt{x^2+y^2}\, dz\, dy\, dx.$$

Solution.

If we actually draw the bounded region in the xy-plane, we see that it is the quarter of the circle of radius 3 in the first quadrant. So we get that our region in cylindrical coordinates is

$$R = \left\{ (r, \theta, z) \, : \, 0 \le r \le 3, \, 0 \le \theta \le \frac{\pi}{2}, \, 0 \le z \le r \right\},$$

hence

$$\begin{aligned} \int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{x^2+y^2}} \sqrt{x^2+y^2}\, dz\, dy\, dx &= \int_0^{\pi/2} \int_0^3 \int_0^r r^2\, dz\, dr\, d\theta \\ &= \int_0^{\pi/2} \int_0^3 r^3\, dr\, d\theta \\ &= \int_0^{\pi/2} \frac{81}{4}\, d\theta = \frac{81}{8}\pi. \end{aligned}$$

Example. Evaluate the integral in spherical coordinates:
$\iiint_D e^{-(x^2+y^2+z^2)}\, dV$, where D is the unit ball.

Solution. The unit ball is $D = \{(x, y, z) \,:\, x^2 + y^2 + z^2 \leq 1\}$, or in spherical coordinates,

$$D = \{(\rho, \phi, \theta) \,:\, 0 \leq \rho \leq 1, 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\}.$$

Hence

$$\begin{aligned}
\iiint_D e^{-(x^2+y^2+z^2)}\, dV &= \int_0^{2\pi} \int_0^{\pi} \int_0^{1} e^{-\rho^2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \int_0^{1} \int_0^{2\pi} \int_0^{\pi} e^{-\rho^2} \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho \\
&= \int_0^{1} \int_0^{2\pi} 2e^{-\rho^2} \rho^2 \, d\theta \, d\rho \\
&= \int_0^{1} 4\pi e^{-\rho^2} \rho^2 \, d\rho \\
&= \text{uh-oh...I think there's a typo in the problem statement}
\end{aligned}$$

Center of Mass in Three Dimensions

Definition. Let ρ be a density function on a closed bounded region $D \subseteq \mathbb{R}^3$. The coordinates of the **center of mass** of the region are $(\overline{x}, \overline{y}, \overline{z})$, where

$$\overline{x} = \frac{M_{yz}}{m} = \frac{1}{m} \iiint_D x\rho(x, y, z)\, dV,$$
$$\overline{y} = \frac{M_{xz}}{m} = \frac{1}{m} \iiint_D y\rho(x, y, z)\, dV,$$
$$\overline{z} = \frac{M_{xy}}{m} = \frac{1}{m} \iiint_D z\rho(x, y, z)\, dV,$$

and $m = \iiint_D \rho(x, y, z)\, dV$ is the mass of the region. M_{xy}, M_{xz}, M_{yz} are the moments with respect to the xy-, xz-, and yz-planes (respectively).

Example. Find the center of mass of the region bounded by the paraboloid $z = 4 - x^2 - y^2$ and $z = 0$ with density $\rho(x, y, z) = 5 - z$.

Solution.

It might be easier to do this in cylindrical coordinates. We have that $D = \{(r, \theta, z) \,|\, 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi,\ 0 \leq z \leq 4 - r^2\}$ and then $\rho(r, \theta, z) = 5 - z$. So

$$\begin{aligned} m &= \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} (5 - z) r \, dz \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 -\frac{1}{2} r^5 - r^3 + 12r \, dr \, d\theta \\ &= \int_0^{2\pi} \frac{44}{3} \, d\theta = \frac{88}{3}\pi. \end{aligned}$$

Using the substitution $x = r \sin\theta$, we then have that

$$\begin{aligned} \overline{x} &= \frac{3}{44\pi} \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} r^2 \sin\theta (5 - z) \, dz \, dr \, d\theta \\ &= stuff \end{aligned}$$

Change of coordinates is useful because regions are ill-behaved in the real world, but they might be just minor modifications of relatively nice regions. This is more-or-less the underlying principle of differential geometry.

Definition. Let $x = g(u, v)$ and $y = h(u, v)$ be differentiable on a region of the uv-plane. The **Jacobian (determinant)** is

$$J(u, v) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Theorem 1. *Suppose R is a region in the xy-plane and S is a region in the uv-plane. Suppose further that $x = g(u,v)$ and $y = h(u,v)$ is a one-to-one transformation taking S to R, and that g, h both have continuous first partial derivatives in S. If f is continuous on R, then*

$$\iint_R f(x,y)\,dA = \iint_S f(g(u,v), h(u,v))|J(u,v)|\,dA.$$

The 3-dimensional analogue of this theorem is exactly what you would expect it to be, and results in

$$\iiint_R f(x,y,z)\,dV = \iiint_S f(g(u,v,w), h(u,v,w), k(u,v,w))|J(u,v,w)|\,dV.$$

Example. Let R be the region bounded by $x+y=1$, $x-y=1$, $x+y=3$, $x-y=-1$. Use a change of coordinates to evaluate the integral $\displaystyle\iint_R (x+y)^2 \sin^2(x-y)\,dA$

Solutions.

Let $u = x+y$ and $v = x-y$. Then we have that $1 \le u \le 3$ and $-1 \le v \le 1$. As well, $x = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(u-v)$, so $J(u,v) = -\frac{1}{2}$. Thus

$$\begin{aligned}
\iint_R (x+y)^2 \sin^2(x-y)\,dA &= \frac{1}{2}\iint_S u^2 \sin^2 v\,dA \\
&= \frac{1}{2}\int_{-1}^{1}\int_{1}^{3} u^2 \sin^2 v\,du\,dv \\
&= \frac{13}{3}\int_{-1}^{1} \sin^2 v\,du\,dv \\
&= \frac{13}{6}\left(2-\sin(2)\right).
\end{aligned}$$

Assignment

Worksheet 12:

`https://mathpost.asu.edu/~wells/math/teaching/mat272_spring2015/homework12.pdf`

As always, you may work in groups, but every member must individually submit a homework assignment.