Recitation 11: Double Integrals in Polar Coordinates & Triple Integrals

Joseph Wells
Arizona State University

April 5, 2015

Sometimes polar coordinates are just nicer to work with. Suppose $f(r, \theta)$ is a function of polar coordinates and R is the region $R = \{(r, \theta) \,|\, a \leq r \leq b,\ \alpha \leq \theta \leq \beta\}$. Then the volume under $f(r, \theta)$ is given by

$$\iint_R f(r, \theta)\, dA = \int_\alpha^\beta \int_a^b f(r, \theta)\, r\, dr\, d\theta.$$

More generally, if r is bounded by some (nonnegative) functions $g(\theta)$ and $h(\theta)$, we have that $R = \{(r, \theta) \,|\, g(\theta) \leq r \leq h(\theta),\ \alpha \leq \theta \leq \beta\}$ and the volume under $f(r, \theta)$ is given by

$$\iint_R f(r, \theta)\, dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r, \theta)\, r\, dr\, d\theta.$$

Example. Find the volume of the solid bounded between the paraboloids $f(x, y) = x^2 + y^2$ and $g(x, y) = 2 - x^2 - y^2$.

Solution. In polar coordinates $r^2 = x^2 + y^2$, so our two paraboloids are given by $\tilde{f}(r, \theta) = r^2$ and $\tilde{g}(r, \theta) = 2 - r^2$. These paraboloids meet along the circle $r = 1$, so our region is given by $R = \{(r, \theta) \,|\, 0 \leq r \leq 1 : 0 \leq \theta \leq 2\pi\}$. Hence

$$\begin{aligned}
\iint_R \tilde{g}(r, \theta) - \tilde{f}(r, \theta)\, dA &= \int_0^{2\pi} \int_0^1 (2 - 2r^2) r \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^1 2r - 2r^3 \, dr \, d\theta \\
&= \int_0^{2\pi} \left[r^2 - \frac{1}{2} r^4 \right]_0^1 d\theta \\
&= \int_0^{2\pi} \frac{1}{2} \, d\theta \\
&= \left[\frac{1}{2} \theta \right]_0^{2\pi} = \pi.
\end{aligned}$$

Example. A thin plate has a density given by $\rho(r,\theta) = 4 + r\sin\theta\,\text{kg/m}^2$. Find the mass of the thin half annulus represented by the region $R = \{(r,\theta) \mid 1 \le r \le 4,\ 0 \le \theta \le \pi\}$.

Solution.

$$
\begin{aligned}
M = \iint_R \rho(r,\theta)\,dA &= \int_0^\pi \int_1^4 (4 + r\sin\theta)r\,dr\,d\theta \\
&= \int_0^\pi \int_1^4 4r + r^2\sin\theta\,dr\,d\theta \\
&= \int_0^\pi \left[2r^2 + \frac{1}{3}r^3\sin\theta\right]_1^4 d\theta \\
&= \int_0^\pi 30 + 21\sin\theta\,d\theta \\
&= [30\theta - 21\cos\theta]_0^\pi = 30\pi + 42
\end{aligned}
$$

Example. Find the volume of the region between the sphere $x^2 + y^2 + z^2 = 19$ and the hyperboloid $z^2 - x^2 - y^2 = 1$, for $z > 0$.

Solution. **Inner integral with respect to z:** For any values x, y, we have that $x^2 + y^2 + 1 \leq z^2 \leq 19 - x^2 - y^2$, so $\sqrt{x^2 + y^2 + 1} \leq z \leq \sqrt{19 - x^2 - y^2}$.

Outer integrals with respect to our region: The two surfaces intersect along the circle $x^2 + y^2 = 9$, so our region is $R = \{(x, y) \,|\, x^2 + y^2 \leq 9\}$, which can be rewritten as $R = \{(x, y) \mid -\sqrt{9 - y^2} \leq x \leq \sqrt{9 - y^2}, -3 \leq y \leq 3\}$. As such, our integral becomes

$$\iiint_R dV = \int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2+1}}^{\sqrt{19-x^2-y^2}} dz\, dx\, dy$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \sqrt{19 - x^2 - y^2} - \sqrt{x^2 + y^2 + 1}\, dz\, dx\, dy.$$

Because we can rewrite our region in polar coordinates as $R = \{(r, \theta) \mid 0 \le r \le 3,\ 0 \le \theta \le 2\pi\}$, our integral becomes

$$
\begin{aligned}
&= \int_0^{2\pi} \int_0^3 r\sqrt{19 - r^2} - r\sqrt{r^2 + 1}\, dr\, d\theta \\
&= \int_0^{2\pi} \left[-\frac{1}{3}(19 - r^2)^{3/2} - \frac{1}{3}(r^2 + 1)^{3/2} \right]_0^3 d\theta \\
&= \int_0^{2\pi} -\frac{1}{3}10^{3/2} - \frac{1}{3}10^{3/2} + \frac{1}{3}19^{3/2} + \frac{1}{3}\, d\theta \\
&= \int_0^{2\pi} \frac{1}{3}\left(1 + 19\sqrt{19} - 20\sqrt{10}\right) d\theta \\
&= \frac{2\pi}{3}\left(1 + 19\sqrt{19} - 20\sqrt{10}\right).
\end{aligned}
$$

Assignment

Worksheet 11:

`https://mathpost.asu.edu/~wells/math/teaching/mat272_spring2015/homework11.pdf`

As always, you may work in groups, but every member must individually submit a homework assignment.