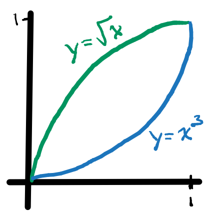
Recitation Notes — Week 10

Last time, we integrated over rectangular regions and saw that Fubini's theorem allowed us to change the order of integration with flagrant disregard. Today we'll integrate over a more general region.

Example 1 $\iint_R 4xy - y^2 \, dA$, where R is the region bounded by $y = \sqrt{x}$, $y = x^3$



We see that we can write our region as $R = \{(x,y) \mid 0 \le x \le 1, \; x^3 \le y \le \sqrt{x}\}$. So,

$$\iint_R 4xy - y^2 \, dA = \int_0^1 \int_{x^3}^{\sqrt{x}} 4xy - y^2 \, dy \, dx$$

$$= \int_0^1 \left[2xy^2 - \frac{1}{3}y^3 \right]_{x^3}^{\sqrt{x}} dx$$

$$= \int_0^1 2x^2 - \frac{1}{3}x^{3/2} - 2x^7 + \frac{1}{3}x^9 \, dx$$

$$= \left[\frac{2}{3}x^3 - \frac{2}{15}x^{5/2} - \frac{1}{4}x^8 + \frac{1}{30}x^{10} \right]_0^1$$

$$= \frac{19}{60}.$$

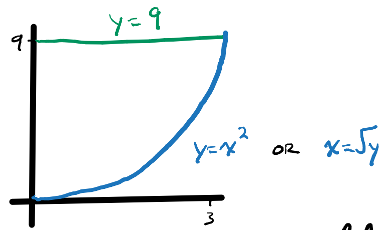
It's not too hard to see that we could have written $R = \{(x,y) \mid \sqrt[3]{y} \le x \le y^2, \; 0 \le y \le 1\}$. This nets us

$$\iint_R 4xy - y^3 \, dA = \int_0^1 \int_{\sqrt[3]{y}}^{y^2} 4xy - y^3 \, dx \, dy.$$

I claim that this integral will net the same solution as the first, but I leave it as an exercise for the reader.

Example Evaluate $\iint_R x^3 e^{y^3}\, dA$ where $R = \{(x,y) \mid 0 \le x \le 3,\ x^2 \le y \le 9\}$

We see that integrating first with respect to y causes a problem as we're missing an extra y^2 term in our integrand. We'll try integrating first with respect to x instead.



We can rewrite our region as $R = \{(x,y) \mid 0 \le x \le \sqrt{y},\ 0 \le y \le 9\}$. Then

$$\iint_R x^3 e^{y^3}\, dA = \int_0^9 \int_0^{\sqrt{y}} x^3 e^{y^3}\, dx\, dy$$

$$= \int_0^9 \left[\frac{1}{4} x^4 e^{y^3}\right]_0^{\sqrt{y}} dy$$

$$= \int_0^9 \frac{1}{4} y^2 e^{y^3}\, dy \qquad u = y^3,\ du = 3y^2 dy$$

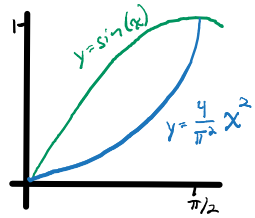
$$= \int_{u=0}^{u=729} \frac{1}{12} e^u\, du$$

$$= \left[\frac{1}{12} e^u\right]_0^{729}$$

$$= \frac{1}{12}\left(e^{729} - 1\right).$$

Let's pretend there are (spatial) units on our function. Every time we integrate, we increase the (spatial) dimension by 1. This means that a double integral calculates volume, which is 3-dimensional. So, it may sound weird that we can use double integration to calculate area. Of course, an object with base area A and height 1 will have volume $V = 1A = A$. So, to determine area, we can take our surface to be defined by $f(x,y) = 1$.

EXAMPLE Use double integration to find the area between $y = \sin(x)$ and $y = \frac{4}{\pi^2}x^2$.



Let $f(x,y) = 1$. We can determine our region $R = \{(x,y) \mid 0 \le x \le \pi/2,\ \frac{4}{\pi^2}x^2 \le y \le \sin(x)\}$. Then

$$A = \iint_R f(x,y)\,dA = \int_0^{\pi/2}\int_{4x^2/\pi^2}^{\sin(x)} dy\,dx$$

$$= \int_0^{\pi/2} \Big[y\Big]_{4x^2/\pi^2}^{\sin(x)} dx$$

$$= \int_0^{\pi/2} \left(\sin(x) - \frac{4}{\pi^2}x^2\right) dx$$

$$= \left[-\cos(x) - \frac{4}{3\pi^2}x^3\right]_0^{\pi/2}$$

$$= \frac{\pi}{6} + 1.$$