Recitation 03: Curves and Planes in Space; Calculus of Vector-Valued Functions

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When we say a function is "[blank]-valued", we mean that the output of the function is [blank]. So "real-valued" functions output real numbers, "integral-valued" functions output integers, and "vector-valued" functions output vectors. So, if given a function

$$\mathbf{f}(t) = \langle f_1(t), f_2(t), \ldots, f_n(t) \rangle,$$

we say that $\mathbf{f}(t)$ is a vector-valued function and the functions $f_i(t)$ are called the *component functions*.

The equation of a line passing through the point $P_0 = (x_0, y_0, z_0)$ in the direction of the vector $\mathbf{v} = \langle a, b, c\rangle$ is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, or

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle,$$

where $-\infty < t < \infty$. Equivalently, the parametric equations for this line are

$$x = x_0 + ta \qquad y = y_0 + tb \qquad z = z_0 + tc.$$

$\mathbf{v}$ above is the *direction vector.* We say that two lines are *parallel* if they have the same direction vector. We say that two lines *intersect* if they are equal for some value of t, and we say that they are *skew* if they are neither parallel nor intersecting.

Example. Find the equation of the line through $P = (0, 4, 8)$ and $Q = (10, -5, -4)$.

Solution

To use our formula, we need a vector, so $\overrightarrow{PQ} = \langle 10, -9, -12 \rangle$. Our formula then says that the equation of the line is given by

$$\mathbf{r} = P + t\overrightarrow{PQ} = \langle 0, 4, 8 \rangle + t\langle 10, -9, -12 \rangle = \langle 10t, 4 - 9t, 8 - 12t \rangle.$$

We can also take limits of vector-valued functions. Suppose $\mathbf{r}(t) = \langle r_1(t), \ldots, r_n(t) \rangle$ is a vector-valued function. Then for some fixed real number a,

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} r_1(t), \ldots, \lim_{t \to a} r_n(t) \right\rangle.$$

Example. Is the following function continuous at $t = 1$? $f(t) = \left\langle \frac{1}{t-1}, t-1 \right\rangle$.

Recall that a function is continuous at a point if the limit exists at that point and the function at that point agrees with the limit. Since $\lim_{t \to 1} \frac{1}{t-1}$ does not exist, $f(t)$ is not continuous at $t = 1$.

Given a vector-valued function $\mathbf{r}(t) = \langle r_1(t), \ldots, r_n(t) \rangle$, the derivative is given by

$$\mathbf{r}'(t) = \frac{d}{dt}\mathbf{r}(t) = \left\langle \frac{d}{dt}r_1(t), \ldots, \frac{d}{dt}r_n(t) \right\rangle = \langle r_1'(t), \ldots, r_n'(t) \rangle.$$

Provided $r'(t) \neq 0$ at the point t, then $r'(t)$ is the *tangent vector* at the point t. The *unit tangent vector* is then

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

The same derivative rules have natural analogs for vector-valued functions. They are:

$$\text{constant rule}: \frac{d}{dt}\left[\mathbf{c}\right] = 0$$

$$\text{sum rule}: \frac{d}{dt}\left[\mathbf{u}(t) + \mathbf{v}(t)\right] = \mathbf{u}'(t) + \mathbf{v}'(t)$$

$$\text{product rule}: \frac{d}{dt}\left[f(t)\mathbf{u}(t)\right] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$$

$$\text{chain rule}: \frac{d}{dt}\left[\mathbf{u}(f(t))\right] = \mathbf{u}'(f(t))f'(t)$$

$$\text{dot product rule}: \frac{d}{dt}\left[\mathbf{u}(t) \cdot \mathbf{v}(t)\right] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$$

$$\text{cross product rule}: \frac{d}{dt}\left[\mathbf{u}(t) \times \mathbf{v}(t)\right] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$$

Example.

We can also take integrals of vector-valued functions.

$$\int_a^b \mathbf{r}(t)\,dt = \left\langle \int_a^b r_1(t)\,dt, \ldots, \int_a^b r_n(t)\,dt \right\rangle.$$

If the integral is indefinite, we still have a $+\mathbf{C}$ at the end, but this time $\mathbf{C}$ is a constant vector.

Example.

Assignment

Worksheet 03:

`https://mathpost.asu.edu/~wells/math/teaching/mat272_spring2015/homework03.pdf`

As always, you may work in groups, but every member must individually submit a homework assignment.