Recitation 02: Dot Product and Cross Product

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Given vectors $\mathbf{a} = \langle a_1, \ldots, a_n \rangle$ and $\mathbf{b} = \langle b_1, \ldots, b_n \rangle$, the *dot product* or *(Euclidean) inner product* is given by

$$a \cdot b = a_1 b_1 + \cdots + a_n b_n = |a||b| \cos(\theta),$$

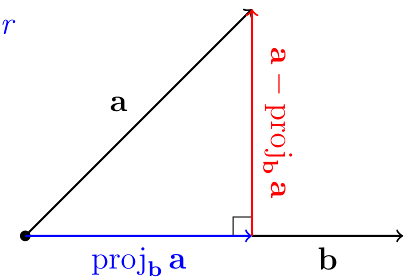
where θ is the angle between the two vectors. The *vector projection of* $\mathbf{a}$ *on* $\mathbf{b}$ is given by

$$\operatorname{proj}_{\mathbf{b}} \mathbf{a} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2} \right) \mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \right) \mathbf{b}.$$

and the *vector rejection of* $\mathbf{a}$ *on* $\mathbf{b}$ is given by

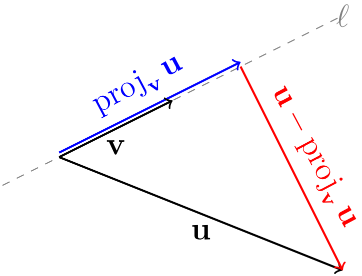
$$\mathbf{a} - \operatorname{proj}_{\mathbf{b}} \mathbf{a}.$$

The vector rejection is the vector that represents the shortest path between the head of vector $\mathbf{a}$ and the vector $\mathbf{b}$.



Example. Find the distance between the point $R = (5, 1)$ and the line ℓ given by $y = \frac{1}{2}x + 3$.

Let's pick two random points on the line, say $P = (0, 3)$ and $Q = (2, 4)$. We can then form vectors $\mathbf{u} = \overrightarrow{PR} = \langle 5, -2 \rangle$ and $\mathbf{v} = \overrightarrow{PQ} = \langle 2, 1 \rangle$. To find the distance between R and the line is the same as finding the length of the vector rejection of $\mathbf{u}$ on $\mathbf{v}$. So



$$\begin{aligned} \operatorname{proj}_{\mathbf{v}} \mathbf{u} &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} \\ &= \left(\frac{8}{5} \right) \langle 2, 1 \rangle, \end{aligned}$$

whence

$$|\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u}| = \left| \langle 5, -2 \rangle - \left(\frac{8}{5} \right) \langle 2, 1 \rangle \right| = \frac{9}{\sqrt{5}} \approx 4.025.$$

In physics, work is computed using a force vector, $\mathbf{F}$, and a distance vector, $\mathbf{d}$, via

$$W = \mathbf{F} \cdot \mathbf{d} = |\mathbf{F}||\mathbf{d}| \cos \theta,$$

where θ is the angle between the force vector and distance vector. Note that work is a *SCALAR*, not a vector.

Example. *No Example at this time.*

Rewriting our vectors as $\mathbf{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\mathbf{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$, the *cross product* is given by

$$\begin{aligned} a \times b &= \det \begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \\ &= (a_2b_3 - a_3b_2)\hat{\imath} - (a_1b_3 - a_3b_1)\hat{\jmath} + (a_1b_2 - a_2b_1)\hat{k}. \end{aligned}$$

Note: the cross product *ONLY WORKS IN 3 DIMENSIONS*.

The length of the cross product vector is given by the usual vector length formula, but can also be given in terms of the angle θ between the vectors by

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta.$$

In physics, torque is computed using a force vector, $\mathbf{F}$, and a radius vector, $\mathbf{r}$, via

$$\tau = \mathbf{r} \times \mathbf{F}.$$

This means that torque is a *VECTOR*.

Cross products also have applications to parallelograms and parallelapipeds. Specifically, for a parallelogram formed by vectors $\mathbf{a}$ and $\mathbf{b}$, the area of the parallelogram is

$$Area = |\mathbf{a} \times \mathbf{b}|.$$

For a parallelapiped formed by vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, the volume of the parallelapiped is

$$Vol = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|.$$

Assignment

Worksheet 02:

https://mathpost.asu.edu/~wells/math/teaching/mat272_spring2015/homework02.pdf

As always, you may work in groups, but every member must individually submit a homework assignment.