Recitation 14: Parametric Equations

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Example (Book §10.1, # 11 and # 12).

a. Eliminate the Parameter to obtain an equation in x and y.

b. Describe the curve and indicate the positive orientation

11. $x = \sqrt{t} + 4,\ y = 3\sqrt{t};\ 0 \le t \le 16.$

12. $x = (t+1)^2,\ y = t+2;\ -10 \le 10 \le 10$

11. Let $\sqrt{t} = \frac{y}{3}$. Then

$$x = \sqrt{t} + 4 = \frac{y}{3} + 4$$
$$\Rightarrow y = 3x - 4.$$

This is just a line.

12. Let $y - 1 = t + 1$. Then

$$\begin{aligned} x &= (t+1)^2 \\ &= (y-1)^2 \end{aligned}$$

This is just a (horizontal) parabola.

Taking a derivative is easy in parametric equations too. Notice that $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

Example (Book §10.1, #47)**.**

a. Determine $\frac{dy}{dx}$ in terms of t and evaluate it at the given value of t.

b. Make a sketch of the curve showing the tangent line at the point corresponding to the given value of t.

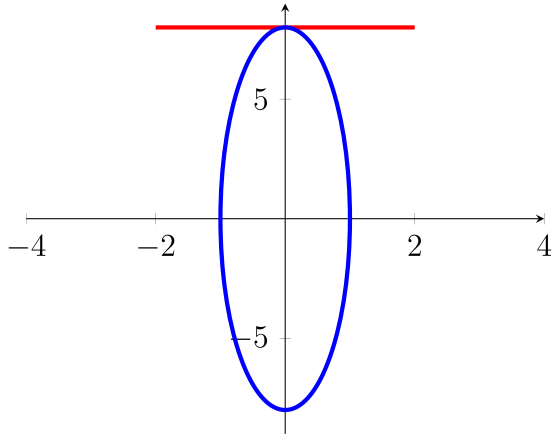
$x = \cos(t)$, $y = 8\sin(t)$, $t = \frac{\pi}{2}$.

By our previous equation

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{8\cos(t)}{-\sin(t)}$$

so

$$\left.\frac{dy}{dx}\right|_{t=\pi/2} = \frac{8\cos\left(\frac{\pi}{2}\right)}{-\sin\left(\frac{\pi}{2}\right)} = 0$$


5
−4
−2
2
4
−5

Theorem. *If a smooth curve C is given by $x = f(t)$ and $t = g(t)$ such that C does not intersect itself on the interval $a \leq t \leq b$ (except possibly at the endpoints), then the arc length of C over the interval is given by*

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$$

Example. Suppose we want to find the length of the curve C from $t = 0$ to $t = \frac{\pi}{2}$: $x = 5\cos(t) - \cos(5t)$, $y = 5\sin(t) - \sin(5t)$.

$$\begin{aligned}
s &= \int_0^{\pi/2} \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt \\
&= \int_0^{\pi/2} \sqrt{[-5\sin(t) + 5\sin(5t)]^2 + [5\cos(t) - 5\cos(5t)]^2}\, dt \\
&= \int_0^{\pi/2} \sqrt{[5(-\sin(t) + \sin(5t))]^2 + [5(\cos(t) - \cos(5t))]^2}\, dt \\
&= 5\int_0^{\pi/2} \sqrt{[-\sin(t) + \sin(5t)]^2 + [\cos(t) - \cos(5t)]^2}\, dt \\
&= 5\int_0^{\pi/2} \sqrt{2 - 2\sin(t)\sin(5t) - 2\cos(t)\cos(5t)}\, dt \\
&= 5\int_0^{\pi/2} \sqrt{2 - 2\cos(4t)}\, dt
\end{aligned}$$

$$= 5\int_0^{\pi/2} \sqrt{4\sin^2(2t)}\,dt$$

$$= 10\int_0^{\pi/2} \sin(2t)\,dt$$

$$= -5\left[\cos(2t)\right]_0^{\pi/2} = 10.$$

Theorem. *If a smooth curve C given by $x = f(t)$ and $y = g(t)$ does not cross itself on an interval $a \leq t \leq b$, then the area S of the surface of revolution formed by revolving C about the coordinate axes is given by the following*

$$S = 2\pi \int_a^b g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \textit{(x-axis)}$$

$$S = 2\pi \int_a^b f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \textit{(y-axis).}$$

Example. Let C be the arc of the circle $x^2 + y^2 = 9$ from $(3, 0)$ to $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$. Find the area of the surface formed by revolving C about the x-axis.

Note that we can represent C parametrically by the equations $x = 3\cos(t)$, $y = 3\sin(t)$, $0 \leq t \leq \frac{\pi}{3}$. Then

$$\begin{aligned}
S &= 2\pi \int_0^{\pi/3} 3\sin(t)\sqrt{[-3\sin(t)]^2 + [3\cos(t)]^2}\, dt \\
&= 6\pi \int_0^{\pi/3} \sin(t)\sqrt{9\left[\sin^2(t) + \cos^2(t)\right]}\, dt \\
&= 6\pi \int_0^{\pi/3} \sin(t)\, dt \\
&= -18\pi\left[\cos(t)\right]_0^{\pi/3} \\
&= 9\pi.
\end{aligned}$$

Assignment

Due (Wed) April 30 / (Fri) May 2

Recitation Notebook:
§9.3 - # 2
§9.4 - # 2
§10.2 - # 2
§10.3 - # 3
§10.4 - # 4

As always, you may work in groups, but every member must individually submit a homework assignment.