Recitation 12: Alternating & Power Series

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Theorem (Alternating Series Test). *The alternating series* $\sum(-1)^{k+1}a_k$ *converges if*

- $a_k \geq a_{k+1} > 0$ *for all* $k > N$, *where* N *is sufficiently large, and*
- $\lim_{k\to\infty} a_k = 0.$

Definition. Given a series $\sum a_n$, we say that $\sum a_n$ *converges absolutely* if $\sum |a_n|$ converges. We say that it *converges conditionally* if $\sum a_n$ converges but $\sum |a_n|$ does not.

Example (Rec Ntbk §8.6, #1b). $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k^2+10}$

Let $a_k = \frac{1}{k^2+10}$. Then we have that, $a_k \geq a_{k+1} > 0$ and $\lim_{k\infty} a_k = 0$, so it converges by the Alternating Series Test

Definition. A *power series* can effectively be thought of as an "infinite polynomial", and is of the form $\sum_{k=0}^{\infty} c_k(x-a)^k$, where a is a constant called the *center* of the series.

Definition. A *Taylor series* is a power series whose coefficients are $c_k = \frac{f^{(k)}(a)}{k!}$. A *Taylor polynomial* is just some finite portion of a Taylor series (that is $k = 0, \ldots, n$).

Example (Rec Ntbk §9.1, #1).

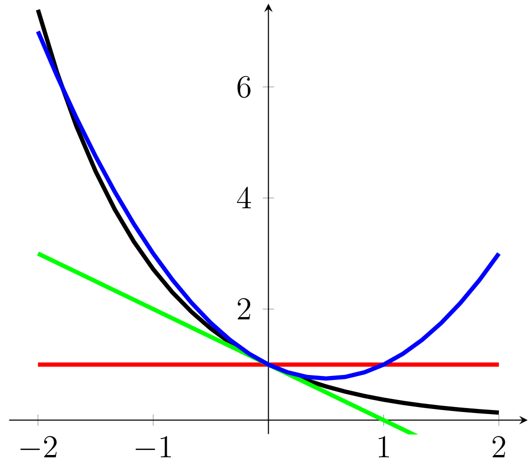
a. Find the n^{th} order Taylor polynomial for $f(x) = e^{-x}$ centered at 0 for $n = 0, 1, 2$.

b. Graph the Taylor polynomials and the function.

a.

$$\begin{aligned} {\color{red}p_0(x)} &= \frac{f(0)}{0!} \\ &= 1 \\ {\color{green}p_1(x)} &= \frac{f(0)}{0!} + \frac{f'(0)}{1!}(x-0)^1 \\ &= 1 - x \\ {\color{blue}p_2(x)} &= \frac{f(0)}{0!} + \frac{f'(0)}{1!}(x-0)^1 + \frac{f'(0)}{2!}(x-0)^2 \\ &= 1 - x + \frac{1}{2}x \end{aligned}$$

b.



Example (Book §9.1, #33).

a. Approximate $e^{0.12}$ using the n^{th} Taylor polynomail with $n = 3$.

b. Compute the absolute error in approximation assuming the exact value is given by a calculator.

Solution.

a. Since 0.12 is close to 0, we choose to center this Taylor series at 0. Then $p_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$, so $p_3(.12) \approx 1.127488$.

b. $e^{0.12} - p_3(0.12) \approx 1.127496852 - 1.127488 = 8.852 \times 10^{-6}$

Definition. Given a power series $\sum c_k(x-a)^k$, the *interval of convergence* is the set of all x such that the series converges. The *radius of convergence* is the distance from the center of the series to the boundary of the interval of convergence.

Example (Rec Ntbk §9.2, #2)**.** Determine the radius of convergence of $\sum \left(-\frac{x}{10}\right)^{2k}$. Then test the endpoints to determine the interval of convergence.

$$\sum \left(-\frac{x}{10}\right)^{2k} = \sum \left(\frac{x^2}{100}\right)^k$$

So by the Root test,

$$\lim_{k\to\infty} \sqrt[k]{\left(\frac{x^2}{100}\right)^k} = \lim_{k\to\infty} \frac{x^2}{100} = \frac{x^2}{100}$$

So this converges whenever $|x| < 10$. Thus, the radius of convergence is 10 and since the series is centered at 0, the interval of convergence is $(-10, 10)$.

Example (Book §9.2, #33)**.** Find the power series representation for $g(x) = \frac{1}{(1-x)^2}$ centered at 0 by differentiating or integrating the power series for $f(x) = \frac{1}{1-x}$ (perhaps more than once). Given the interval of convergence for the resulting series.

The power series for $f(x)$ is $\displaystyle\sum_{k=0}^{\infty} x^k$, which is convergent for $x \in (-1, 1)$. So the power series for $g(x) = f'(x)$ is $\displaystyle\sum_{k=1}^{\infty} kx^{k-1} = \sum_{k=0}^{\infty} (k+1)x^k$, which is also convergent for $x \in (-1, 1)$. So the interval of convergence is $(-1, 1)$.

Assignment

Recitation Notebook:
§8.6 - #2, #6
§9.1 - #2, #3
§9.2 - #1, #4

As always, you may work in groups, but every member must individually submit a homework assignment.