Recitation 05: Improper Integrals & Application of Integration

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I would like to start today with a short story to demonstrate why it's important to have calculus in your repertoire. In the 1990's, a researcher, Dr. Mary Tai, was studying glucose tolerances (and the like) as part of her research on diabetes, when she thought she struck gold. In 1994, she published a paper entitled "A Mathematical Model for the Determination of Total Area Under Glucose Tolerance and Other Metabolic Curves". Her particular method was a numerical method akin to Riemann sums, called the trapezoid rule, which is a set of equations called the Newton-Cotes equations dating back to the mid-1700s.

Miraculously, Dr. Tai procured funding to reinvent calculus, and her paper managed to pass through editing, peer review, and the hands of a Yale professor of engineering before making it's way to a journal of diabetes.

So, why is calculus important? Not only does it save time, but it saves you from potential ridicule from the entire scientific community for reinventing the wheel.

Example. Evaluate the following integral, or state that it diverges $\displaystyle\int_2^\infty \frac{x}{(x+2)^2}\,dx$.

Solution.

We'll use the substitution $u = x + 2$ annd $du = dx$. Then

$$\begin{aligned}
\int_2^\infty \frac{x}{(x+2)^2}\,dx &= \lim_{t\to\infty} \int_{x=2}^{x=t} \frac{x}{(x+2)^2}\,dx \\
&= \lim_{t\to\infty} \int_{x=2}^{x=t} \frac{u-2}{u^2}\,du \\
&= \lim_{t\to\infty} \int_{x=2}^{x=t} \left[\frac{1}{u} - \frac{2}{u^2}\right]\,du \\
&= \lim_{t\to\infty} \left[\ln(u) + \frac{2}{u}\right]_{x=2}^{x=t} \\
&= \lim_{t\to\infty} \left[\ln(x+2) + \frac{2}{x+2}\right]_{x=2}^{x=t}
\end{aligned}$$

$$= \lim_{t\to\infty} \left[\ln(t+2) + \frac{2}{t+2} - \ln(4) - \frac{1}{2} \right]$$

$$= \lim_{t\to\infty} \left[\frac{(t+2)\ln(t+2)+2}{t+2} - \ln(4) - \frac{1}{2} \right]$$

Since the two terms on the right are constants, we care only about what's happening to the left term.

$$\lim_{t\to\infty} \frac{(t+2)\ln(t+2)+2}{t+2} \stackrel{L'H}{=} \lim_{t\to\infty} \frac{\ln(t+2)+1}{1} = \infty$$

Thus the integral diverges.

Example. Use symmetry to evaluate the following integral $\displaystyle\int_{-\infty}^{\infty} e^{-|t|}\,dt$.

Solution.

$$
\begin{aligned}
\int_{-\infty}^{\infty} e^{-|t|}\,dt &= 2\int_{0}^{\infty} e^{-t}\,dt \\
&= \lim_{n\to\infty} 2\int_{0}^{n} e^{-t}\,dt \\
&= \lim_{n\to\infty} -2e^{-t}\Big|_{0}^{n} \\
&= \lim_{n\to\infty} 2-2e^{-n} \\
&= 2.
\end{aligned}
$$

Example. Let's say we're interested in blasting a 100 kg ($\approx$ 220 lbs) classmate off into space. How much work has to be done to escape Earth's gravity?

Solution.

Following from the universal law of gravitation, the amount of work required to escape Earth's gravity is given by

$$\int_{r_0}^{\infty} \frac{Gm_1m_2}{r^2}\,dr,$$

where

$$\begin{aligned}
r_0 &= 6.38 \times 10^6\,\text{m (the approximate radius of Earth)}\\
G &= 6.67 \times 10^{-11}\,\text{N} \cdot \text{m}^2/\text{kg}^2 \text{ (the Universal Gravitation Constant)}\\
m_1 &= 100\,\text{kg (the mass of your classmate)}\\
m_2 &= 6 \times 10^{24}\,\text{kg (the mass of Earth)}.
\end{aligned}$$

So since this integral is improper, we use limits to see that

$$\begin{aligned}
\int_{r_0}^{\infty} \frac{Gm_1m_2}{r^2}\,dr &= Gm_1m_2 \int_{r_0}^{\infty} \frac{1}{r^2}\,dr \\
&= Gm_1m_2 \lim_{t\to\infty} \int_{r_0}^{t} \frac{dr}{r^2} \\
&= Gm_1m_2 \lim_{t\to\infty} \left[-\frac{1}{r}\right]_{r_0}^{t} \\
&= Gm_1m_2 \lim_{t\to\infty} \left[\frac{1}{r_0} - \frac{1}{t}\right] \\
&= \frac{Gm_1m_2}{r_0} \\
&\approx 6.292 \times 10^9\,\mathrm{N}\cdot\mathrm{m} = 6.292 \times 10^9\,\mathrm{J}.
\end{aligned}$$

Assignment

Recitation Notebook:
§7.7 - #1, #2, #3
and the following worksheet:
`http://math.joedub.net/teaching/mat271_spring2014/mat271_spring2014_homework05.pdf`

As always, you may work in groups, but every member must individually submit a homework assignment.