# Recitation 02: Integration by Parts & Trig Integrals

Joseph Wells
Arizona State University

January 21, 2014

Recall the theorems:

**Theorem** (Integration by Parts)**.** *Suppose that* $u$ *and* $v$ *are differentiable functions. Then*

$$\int u\,dv = uv - \int v\,du$$

**Theorem** (Integration by Parts - Definite Integral)**.** *Suppose that* $u$ *and* $v$ *are differentiable on* $(a,b)$*. Then*

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\,dx$$

**Example** (Integration by Parts)**.** Evaluate the following integral:

$$\int \theta \sec^2(\theta)\, d\theta$$

*Solution.*

Let $u = \theta$ and $dv = \sec^2(\theta)\, d\theta$. Then $du = d\theta$ and $v = \tan(\theta) + C$. So

$$\begin{aligned}\int \theta \sec^2(\theta)\, d\theta &= \theta \tan(\theta) - \int \tan(\theta)\, d\theta \\ &= \theta \tan(\theta) - \ln|\sec(\theta)| + C\end{aligned}$$

**Example** (Repeated Integration by Parts)**.** Evaluate the following integral:

$$\int t^3 e^{-t}\, dt$$

*Solution.*

Since $t^3$ is a polynomial, we should probably pick $u = t^3$ and $dv = e^{-t}$, $dt$. However, doing so will require us to do integration by parts 3 times. Polynomials are actually kind of special in that multiple derivatives will eventually get us to zero. As a result, there is a faster way through it (the Tabular Method).

We form a table with two columns. The left column is $u$ and the right column is $dv$. In the left column, we take a derivative of each preceding row until we get to 0. In the right column, we take an anti-derivative of each preceding row. Our table thus looks like this:

| $u$ | $dv$ |
|---|---|
| $t^3$ | $e^{-t}$ |
| $3t^2$ | $-e^{-t}$ |
| $6t$ | $e^{-t}$ |
| $6$ | $-e^{-t}$ |
| $0$ | $e^{-t}$ |

From here, we alternately assign $\pm$ to the $u$ terms and associate each $u$ to the $dv$ in the row below.

| | $u$ | | $dv$ |
|---|---|---|---|
| $+$ | $t^3$ | $\searrow$ | $e^{-t}$ |
| $-$ | $3t^2$ | $\searrow$ | $-e^{-t}$ |
| $+$ | $6t$ | $\searrow$ | $e^{-t}$ |
| $-$ | $6$ | $\searrow$ | $-e^{-t}$ |
| | $0$ | | $e^{-t}$ |

So then we multiply associated $u$ and $dv$ terms together and add them to get

$$\int t^3 e^{-t}\,dt = -t^3e^{-t} - 3t^2e^{-t} - 6te^{-t} - 6e^{-t} + C.$$

**Example** (Definite Integrals (Integration by Parts))**.** Evaluate the following definite integral:

$$\int_0^{\pi/2} x\cos(2x)\,dx$$

*Solution.*

Let $u = x$ and $dv = \cos(2x)\,dx$. Then $du = dx$ and $v = \frac{1}{2}\sin(2x) + C$, so

$$\begin{aligned}\int_0^{\pi/2} x\cos(2x)\,dx &= \frac{1}{2}x\sin(2x)\Big|_0^{\pi/2} - \frac{1}{2}\int_0^{\pi/2} \sin(2x)\,dx \\ &= \frac{1}{2}x\sin(2x)\Big|_0^{\pi/2} + \frac{1}{4}\cos(2x)\Big|_0^{\pi/2} \\ &= (0-0) + (-\frac{1}{4} - \frac{1}{4}) = -\frac{1}{2}.\end{aligned}$$

**Example** (Integrals of $\sin(x)$ and $\cos(x)$). Evaluate the following integral:

$$\int \sin^2(x)\cos^2(x)\,dx$$

*Solution.*

Since both sin and cos have even powers, we use the half-angle identities to rewrite the integrand.

Recall that $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$ and $\sin^2(x) = \frac{1}{2}(1-\cos(2x)$. So

$$\begin{aligned}
\int \sin^2(x)\cos^2(x) &= \frac{1}{4}\int (1-\cos(2x))\,(1+\cos(2x))\,dx \\
&= \frac{1}{4}\int \left[1-\cos^2(2x)\right]\,dx \\
&= \frac{1}{4}\int \left[1-\frac{1}{2}\,(1+\cos(4x))\right]\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\int dx - \frac{1}{8}\int dx - \frac{1}{8}\int \cos(4x)\,dx \\
&= \frac{1}{4}x - \frac{1}{8}x - \frac{1}{32}\sin(4x) = \frac{x}{8} - \frac{\sin(4x)}{32}.
\end{aligned}
$$

**Example** (Integrals of $\tan(x)$ and $\sec(x)$)**.** Evaluate the following integral:

$$\int \sec^2(x)\tan^{1/2}(x)\,dx$$

*Solution.*

Since sec has an even power, we choose $u = \tan(x)$ and change any leftover $\sec^{2k}(x)$ terms into $(\tan^2(x)+1)^k$. So, with our choice of $u$, we get that $du = \sec^2(x)\,dx$, and thus

$$\begin{aligned}\int \sec^2(x)\tan^{1/2}(x)\,dx &= \int u^{1/2}\,du \\ &= \frac{2}{3}u^{3/2} + C \\ &= \frac{2}{3}\tan^{3/2}(x) + C.\end{aligned}$$

**Example** (Square roots)**.** Evaluate the following integral:

$$\int_0^{\pi/2} \sqrt{1-\cos(2x)}\,dx$$

*Solution.*

Again, we appeal to the half-angle formula $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$ and get

$$\begin{aligned}\int_0^{\pi/2} \sqrt{1-\cos(2x)}\,dx &= \sqrt{2}\int_0^{\pi/2} \sin(x)\,dx \\ &= -\sqrt{2}\cos(x)\Big|_0^{\pi/2} \\ &= \sqrt{2}.\end{aligned}$$

## Assignment

Recitation Notebook:
§7.1 - #1, #2, #6
§7.2 - #1, #3, #4

As always, you may work in groups, but homework must written up and submitted by each individual.