Recitation 14: Parametric Equations & Polar Coordinates

Joseph Wells
Arizona State University

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Example (Rec. Ntbk, §10.1, #1).

a. Eliminate the parameter to obtain an equation in x and y.

b. Describe the curve and indicate the positive orientation

Solution.

i. $x = (t+1)^2$, $y = t+2$; $-10 \leq t \leq 10$

   a. Notice that $t+1 = y-1$, so we have

$$\begin{aligned} x &= (t+1)^2 \\ &= (y-1)^2 \end{aligned}$$

   This is just a (horizontal) parabola that opens to the right and intersects the x-axis at $y = 1$.

   b. The orientation moves up the y-axis, tracing this parabola.

ii. $x = 3\cos(t)$, $y = 3\sin(t)$, $0 \leq t \leq \pi/2$.

a. Using our pythagorean identity, we have

$$\begin{aligned} x^2 + y^2 &= 9(\cos^2(t) + \sin^2(t)) \\ &= 9 \end{aligned}$$

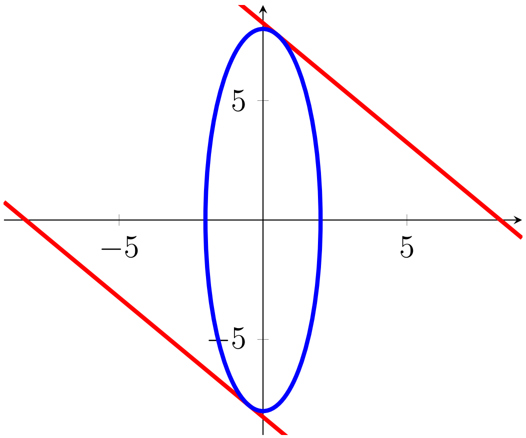
Since $0 \leq t \leq \pi/2$, both $x, y \geq 0$

b. This is a counter-clockwise oriented quarter-circlular arc, centered at the origin with radius 3.

Taking a derivative is easy in parametric equations too. From the chain rule, we have

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}.$$

Example (Rec. Ntbk §10.1, #5)**.** Find all the points on the following curve that has the given slope: $x = 2\cos(t)$, $y = 8\sin(t)$, slope $= -1$.



Solution.

This parameterizes an ellipse with x-radius 2 and y-radius 8. By our previous equation, we solve for t such that

$$\begin{aligned} -1 = \frac{dy}{dx} &= \frac{y'(t)}{x'(t)} \\ &= \frac{8\cos(t)}{-2\sin(t)} \\ \frac{1}{4} &= \frac{\cos(t)}{\sin(t)} \\ 4 &= \tan(t) \\ t &= \arctan(4). \end{aligned}$$

So the points (x, y) where the slope is -1 are

$$(x, y) = \left(\pm\frac{2}{\sqrt{17}}, \pm\frac{32}{\sqrt{17}}\right).$$

To convert from Cartesian coordinates (x, y) to polar coordinates (r, θ), we use the conversions below

$$x = r\cos(\theta), \qquad y = r\sin(\theta),$$

$$r = x^2 + y^2, \qquad \tan(\theta) = \frac{y}{x}.$$

Example (Rec. Ntbk. §10.2, #1)**.** Convert the following equations to Cartesian coordinates. Describe the resulting curve.

a. $r = \cot(\theta)\csc(\theta)$

b. $r = \sin(\theta)\sec^2(\theta)$

Solution.

a.

$$\begin{aligned} r &= \cot(\theta)\csc(\theta) \\ r\sin(\theta) &= \frac{1}{\tan(\theta)} \\ y &= \frac{x}{y} \\ x &= y^2. \end{aligned}$$

The resulting curve is a parabola with vertex $(0, 0)$ that opens to the right.

b.

$$
\begin{aligned}
r &= \sin(\theta)\sec^2(\theta) \\
r\cos(\theta) &= \frac{\sin(\theta)}{\cos(\theta)} \\
x &= \tan(\theta) \\
x &= \frac{y}{x} \\
y &= x^2.
\end{aligned}
$$

The resulting curve is a parabola with vertex $(0,0)$ that opens to the right.

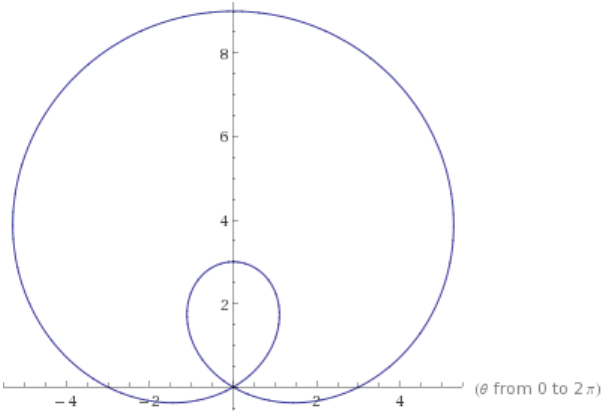
Example (Rec. Ntbk. §10.2, #4)**.** Sketch the following sets of points $\{(r,\theta) : 1 < r < 2,\ \frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}\}$

The shape is a portion of the annulus, where the circular arcs are dashed and the radial lines are solid.

As before, we can also differentiate in polar coordinates. By recognizing that our radius is a function of theta $r = f(\theta)$, we get

$$\frac{dy}{dx} = \frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}.$$

Example (Rec. Ntbk)**.** Find the points (r, θ) for which the polar curve $r = 3 + 6\sin(\theta)$ (called a *limaçon*) has horizontal or vertical tangents.



By our formula, this amounts to solving for pairs (r, θ) for which $f'(\theta)\sin(\theta) + f(\theta)\cos(\theta) = 0$ and $f'(\theta)\cos(\theta) - f(\theta)\sin(\theta) = 0$.

In particular, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{6\cos(\theta)\sin(\theta) + (3+6\sin(\theta))\cos(\theta)}{6\cos(\theta)\cos(\theta) - (3+6\sin(\theta))\sin(\theta)} \\ &= \frac{\cos(\theta)(4\sin(\theta)+1)}{2-\sin(\theta)-4\sin^2(\theta)}.\end{aligned}$$

The numerator is 0 when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, and at the two values of $\theta \in [0, 2\pi]$ for which $\sin(\theta) = -\frac{1}{4}$.

Using the quadratic formula, the denominator is 0 at the two values of $\theta \in [0, 2\pi)$ where $\sin(\theta) = \frac{-1+\sqrt{33}}{8}$ and the two values of $\theta \in [0, 2\pi)$ where $\sin(\theta) = \frac{-1-\sqrt{33}}{8}$.