# Recitation 09: Sequences and Series

Joseph Wells
Arizona State University

October 17, 2014

**Example** (Rec Notebk: §8.2, #3). **Geometric sequences** Determine whether the following sequences converge or diverge and describe whether they do so monotonically or by oscillation. Give the limit when the sequence converges.

**a.** $\{(-1.01)^n\}_{n\in\mathbb{Z}^+}$

**b.** $\{2^n 3^{-n}\}_{n\in\mathbb{Z}^+}$

**a.** $\{(-1.01)^n\}$ diverges since $|-1.01| > 1$. Since it is negative, it oscillates.

**b.** $\{2^n 3^{-n}\} = \{(2\cdot 3^{-1})^n\} = \{(2/3)^n\}$ converges since $|2/3| < 1$. Since it is positive, it does so monotonically. The limit is 0.

**Example** (Book: §8.2, #18). Determine the limit of the following sequence, or state that it diverges: $\{\frac{\ln(1/n)}{n}\}_{n\in\mathbb{Z}^+}$

By L'Hopital's rule,

$$\lim_{n\to\infty}\frac{\ln(1/n)}{n} \overset{L'H}{=} \lim_{n\to\infty}\frac{1}{n} = 0.$$

**Theorem** (8.6). *The following sequences are ordered according to the increasing growth rates as $n \to \infty$. For all positive real numbers $p, q, r, s$ and $b > 1$,*

$$\{\ln^q n\} \ll \{n^p\} \ll \{n^p \ln^r n\} \ll \{n^{p+s}\} \ll \{b^n\} \ll \{n!\} \ll \{n^n\}.$$

**Example** (Rec Notebk: §8.2, #4). **Comparing growth rates of sequences** Determine which sequence has the greater growth rate as $n \to \infty$. Be sure to justify and explain your work: $a_n = 3^n$; $b_n = n!$.

Following from Theorem 8.6, $n! \gg 3^n$ since $n! \gg b^n$ for every $b > 1$.

Recall

**Definition.** A *series* is a limit of sums of terms $b_k$, and is given by $\sum_{k=0}^{\infty} b_k = \lim_{n\to\infty} \sum_{k=0}^{n} b_k$

**Definition.** A *geometric series* is a series of the form $\sum_{k=0}^{\infty} ar^k$ (where $r$ is called the ratio).

If the upper limit of the summation is finite, we have that $\sum_{k=0}^{n} ar^k = \frac{1-r^n}{1-r}$. If we take the limit as $n \to \infty$, we get $\sum_{k=0}^{\infty} ar^k = \lim_{n\to\infty} \sum_{k=0}^{n} ar^k$. What conditions do we need to determine whether or not this limit converges? $|r| < 1$.

**Definition.** A *telescoping series* is a series that has only finitely many terms after cancellation.

**Example** (Book §8.3, #7)**.** Evaluate the following geometric sum: $\sum_{k=0}^{8} 3^k$

$$\sum_{k=0}^{8} 3^k = 1 \cdot \frac{1-3^9}{1-3} = \frac{19682}{2} = 9841$$

**Example** (Book §8.3, #11)**.** Evaluate the following geometric sum: $\sum_{k=0}^{9} \left(-\frac{3}{4}\right)^k$

$$\sum_{k=0}^{9} \left(-\frac{3}{4}\right)^k = 1 \cdot \frac{1-\left(-\frac{3}{4}\right)^{10}}{1+\frac{3}{4}} = \frac{4^{10}-3^{10}}{4^{10}-3\cdot 4^9} = \frac{141,361}{262,144} \approx 0.54$$

**Example** (Book §8.3, #19)**.** Evaluate the geometric series, or state that it diverges: $\sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k$

$\left|\frac{1}{4}\right| < 1$, so it converges to $\displaystyle\sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$.

**Example** (Book §8.3, #29)**.** Evaluate the geometric series, or state that it diverges: $\sum_{k=4}^{\infty} \frac{1}{5^k}$

$\left|\frac{1}{5}\right| < 1$, so it converges to $\displaystyle\sum_{k=4}^{\infty} \frac{1}{5^k} = \frac{\frac{1}{5^4}}{1-\frac{1}{5}} = \frac{1}{5^4-5^3} = \frac{1}{500}$.

**Example** (Book §8.3, #47)**.** Find a formula for the $n$-th term of the sequence of partial sums $\{S_n\}$. Then evaluate $\lim_{n\to\infty} S_n$ to obtain the value of the series or state that the series diverges: $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k+2}\right)$

Notice that

$$\begin{aligned}
S_1 &= \frac{1}{2}-\frac{1}{3}\\
S_2 &= \frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}=\frac{1}{2}-\frac{1}{4}\\
&\vdots\\
S_n &= \frac{1}{2}-\frac{1}{n+2}=\frac{n}{2n+4}.
\end{aligned}$$

Then $\displaystyle\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{n}{2n+4}=\frac{1}{2}.$

**Example** (Book §8.3, #51)**.** Find a formula for the $n$-th term of the sequence of partial sums $\{S_n\}$. Then evaluate $\lim_{n\to\infty} S_n$ to obtain the value of the series or state that the series diverges: $\displaystyle\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k}\right)$

Notice that, $\ln\left(\frac{k+1}{k}\right) = \ln(k+1) - \ln(k)$, so

$$\begin{aligned}
S_1 &= \ln(2) - \ln(1) \\
S_2 &= \ln(2) - \ln(1) + \ln(3) - \ln(2) = \ln(3) - \ln(1) \\
&\vdots \\
S_n &= \ln(n+1) - \ln(1) = \ln(n+1).
\end{aligned}$$

Then $\displaystyle\lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln(n+1)$, which diverges.

**Example** (Book §8.3, #53)**.** Find a formula for the $n$-th term of the sequence of partial sums $\{S_n\}$. Then evaluate $\lim_{n\to\infty} S_n$ to obtain the value of the series or state that the series diverges: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(k+p)(k+p+1)}$, where $p$ is a positive integer.

By partial fractions, $\frac{1}{(k+p)(k+p+1)} = \frac{1}{k+p} - \frac{1}{k+p+1}$, so

$$S_1 = \frac{1}{1+p} - \frac{1}{2+p}$$

$$S_2 = \frac{1}{1+p} - \frac{1}{2+p} + \frac{1}{2+p} - \frac{1}{3+p} = \frac{1}{1+p} - \frac{1}{3+p}$$

$$\vdots$$

$$S_n = \frac{1}{1+p} - \frac{1}{n+p+1} = \frac{n}{(1+p)(n+p+1)} = \frac{n}{n(p+1)+(p+1)^2}.$$

Then $\displaystyle\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n}{n(p+1)+(p+1)^2} = \frac{1}{p+1}.$

## Assignment

Recitation Notebook:
§8.2 - #1, #2
§8.3 - #3, #4, #5
and the following worksheet:
`http://math.joedub.net/teaching/mat271_fall2014/homework09.pdf`

As always, you may work in groups, but every member must individually submit a homework assignment.