MAT271 Exam 1 Review

Fall 2014

This is not a complete list of topics covered in class, but merely a compilation of supplemental exercises from each section. You should still review class notes and the practice exam posted on ASU's MAT271 course page:
`http://math.asu.edu/first-year-math/mat-271-calculus-analytic-geometry-ii`

§5.5 Integration by Substitution

1. $\displaystyle\int \frac{x^2}{16 - x^6}\,dx$

2. $\displaystyle\int (x+1)^9\,dx$

3. $\displaystyle\int_3^5 2x\sqrt{x^2-9}\,dx$

4. $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

§7.1 Integration by Parts

5. $\displaystyle\int e^t \sin(t)\,dt$

6. $\displaystyle\int x \arcsin(2x)\,dx$

7. $\displaystyle\int \ln(\sqrt{x^2-1})\,dx$

8. $\displaystyle\int x\sqrt{x-5}\,dx$

§7.2 Trigonometric Integrals

9. Write the three Pythagorean identities for trigonometric functions.

10. Write down the double angle formulas for $\sin(2x)$ and $\cos(2x)$.

11. Write down the power-reducing formulas for $\sin^2(x)$ and $\cos^2(x)$.

12. $\int \cos^3(\pi x - 1)\, dx$

13. $\int \sec^4\left(\frac{x}{2}\right)\, dx$

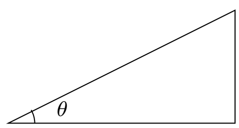
14. $\int \frac{1}{1 - \sin^2(\theta)}\, d\theta$

15. $\int \tan(\theta)\sec^4(\theta)\, d\theta$

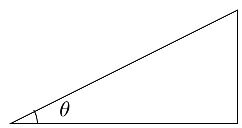
§7.3 Trigonometric Substitutions

Complete the reference triangle for the given trigonometric substitution.

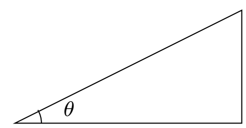
16. $x = a\sin(\theta)$



17. $x = a\sec(\theta)$



18. $x = a\tan(\theta)$



19. $\int \frac{-12}{x^2\sqrt{4 - x^2}}\, dx$

20. $\int \frac{x^3}{\sqrt{4 + x^2}}\, dx$

21. $\int \sqrt{4x^2 - 9}\, dx$

22. $\int_0^{\pi/2} \frac{\sin(\theta)}{1 + 2\cos^2(\theta)}\, d\theta$

§7.4 Partial Fractions

23. From the following equation, write down the augmented coefficient matrix: $8x^4 + 2x^2 + 3x - 9 = (A+B)x^4 + Cx^3 + (D - 2E - A - 3B + C)x^2 + (E - 12D)x + (A + B + C + D - E)$

24. Using a calculator, put the previous matrix into *reduced row echelon form* (RREF).

25. $\int \frac{x^2 - 1}{x^3 + x}\, dx$

26. $\int \frac{x^2 + x + 3}{x^4 + 6x^2 + 9}\, dx$

27. $\int_1^5 \frac{x - 1}{x^2(x + 1)}$

28. $\int \frac{2x^3 - 2x^2 - 4}{(x^2 - x)(x^2 + 4)}\, dx$

§7.7 Improper Integrals

29. $\int_1^\infty \frac{dx}{x}$

30. $\int_0^\infty \frac{1}{x^2 + 1}\, dx$

31. $\int_{-\infty}^\infty \frac{e^x}{1 + e^{2x}}\, dx$

32. $\int_{-1}^1 \frac{1}{x^2}\, dx$

Solutions

1. Let $u = \frac{x^3}{4}$ and $du = \frac{3}{4}x^2\,dx$. Then

$$\begin{aligned}\int \frac{x^2}{16 - x^6}\,dx &= \frac{4}{3}\int \frac{x^2}{16 - 16u^2}\,dx\\ &= \frac{1}{12}\int \frac{x^2}{1-u^2}\,du\\ &= \frac{1}{12}\arcsin(u) + C\\ &= \frac{1}{12}\arcsin\left(\frac{x^3}{4}\right) + C.\end{aligned}$$

2. Let $u = x + 1$ and $du = dx$. Then

$$\begin{aligned}\int (x+1)^9\,dx &= \int u^9\,du\\ &= \frac{1}{10}u^{10} + C\\ &= \frac{1}{10}(x+1)^{10} + C.\end{aligned}$$

3. Let $u = x^2 - 9$ and $du = 2x\,dx$. Then

$$\begin{aligned}\int_{x=3}^{x=5} 2x\sqrt{x^2-9}\,dx &= \int_{u=0}^{u=16}\sqrt{u}\,du\\ &= \frac{2}{3}u^{2/3}\Big|_{u=0}^{u=16}\\ &= \frac{128}{3}.\end{aligned}$$

4. Let $u = e^x + e^{-x}$ and $du = (e^x - e^{-x})\,dx$. Then

$$\begin{aligned}\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx &= \int \frac{1}{u}\,du\\ &= \ln(u) + C\\ &= \ln\left(e^x + e^{-x}\right) + C.\end{aligned}$$

5. Let $u = \sin(t)$ and $dv = e^t\,dt$. Then $du = \cos(t)\,dt$ and $v = e^t$, so

$$\int e^t\sin(t)\,dt = e^t\sin(t) - \int e^t\cos(t)\,dt.$$

Now let $\tilde{u} = \cos(t)$ and $d\tilde{v} = e^t\,dt$. Then $d\tilde{u} = -\sin(t)\,dt$ and $\tilde{v} = e^t$, so

$$\begin{aligned}\int e^t\sin(t)\,dt &= e^t\sin(t) - e^t\cos(t) - \int e^t\sin(t)\,dt\\ 2\int e^t\sin(t)\,dt &= e^t\sin(t) - e^t\cos(t) + C\\ \int e^t\sin(t)\,dt &= \frac{1}{2}e^t\left[\sin(t) - \cos(t)\right] + C.\end{aligned}$$

6. Let $u = \arctan(2x)$ and $dv = x\,dx$. Then $du = \frac{2}{1+4x^2}$ and $v = \frac{1}{2}x^2$, so

$$\int x\arctan(2x)\,dx = \frac{1}{2}x^2\arctan(2x) - \int \frac{x^2}{1+4x^2}\,dx.$$

Making the substitution $x = \frac{1}{2}\tan(\theta)$ and $dx = \frac{1}{2}\sec^2(\theta)\,d\theta$, we get

$$\begin{aligned}\frac{1}{2}x^2\arctan(2x) - \int\frac{x^2}{1+4x^2}\,dx &= \frac{1}{2}x^2\arctan(2x) - \frac{1}{8}\int\frac{\tan^2(\theta)\sec^2(\theta)}{1+\tan^2(\theta)}\,d\theta\\ &= \frac{1}{2}x^2\arctan(2x) - \frac{1}{8}\int\frac{\tan^2(\theta)\sec^2(\theta)}{\sec^2(\theta)}\,d\theta\\ &= \frac{1}{2}x^2\arctan(2x) - \frac{1}{8}\int\tan^2(\theta)\,d\theta\\ &= \frac{1}{2}x^2\arctan(2x) - \frac{1}{8}\int(1+\sec^2(\theta)\,d\theta\\ &= \frac{1}{2}x^2\arctan(2x) - \frac{1}{8}\theta - \frac{1}{8}\tan(\theta) + C\\ &= \frac{1}{2}x^2\arctan(2x) - \frac{1}{8}\arctan(2x) - \frac{1}{4}x + C.\end{aligned}$$

7. Let $u = \ln(\sqrt{x^2-1})$ and $dv = dx$. Then $du = \frac{1}{x^2-1}\,dx$ and $v = x$, so

$$\int \ln(\sqrt{x^2-1})\,dx = x\ln(\sqrt{x^2-1}) - \int \frac{x}{x^2-1}\,dx.$$

Now let $\tilde{u} = \sqrt{x^2-1}$. Then $d\tilde{u} = \frac{x}{\sqrt{x^2-1}}$ so

$$\begin{aligned} x\ln(\sqrt{x^2-1}) - \int \frac{x}{x^2-1}\,dx &= x\ln(\sqrt{x^2-1}) - \int d\tilde{u} \\ &= x\ln(\sqrt{x^2-1}) - \tilde{u} + C \\ &= x\ln(\sqrt{x^2-1}) - \sqrt{x^2-1} + C. \end{aligned}$$

8. Let $u = x$ and $dv = \sqrt{x-5}\,dx$. Then $du = dx$ and $v = \frac{2}{3}(x-5)^{3/2}$, so

$$\begin{aligned} \int x\sqrt{x-5}\,dx &= \frac{2}{3}x(x-5)^{3/2} - \frac{2}{3}\int (x-5)^{3/2}\,dx \\ &= \frac{2}{3}x(x-5)^{3/2} - \frac{4}{15}(x-5)^{5/2} + C. \end{aligned}$$

9. $\sin^2(\theta) + \cos^2(\theta) = 1, \quad \tan^2(\theta) + 1 = \sec^2(\theta), \quad 1 + \cot^2(\theta) = \csc^2(\theta)$

10. $\sin(2\theta) = 2\sin(\theta)\cos(\theta), \quad \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$

11. $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}, \quad \cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$

12. To simplify things a bit, let $u = \pi x - 1$. Then $du = \pi\,dx$, so along with the Pythagorean identity we get

$$\begin{aligned} \int \cos^3(\pi x - 1)\,dx &= \frac{1}{\pi}\int \cos^3(u)\,du \\ &= \frac{1}{\pi}\int (1 - \sin^2(u))\cos(u)\,du \\ &= \frac{1}{\pi}\int \cos(u)\,du - \frac{1}{\pi}\int \sin^2(u)\cos(u)\,du. \end{aligned}$$

We make the substitution $v = \sin(u)$, $dv = \cos(u)\,du$ leading us to

$$\begin{aligned} \frac{1}{\pi}\int \cos(u)\,du - \frac{1}{\pi}\int \sin^2(u)\cos(u)\,du &= \frac{1}{\pi}\sin(u) - \frac{1}{3\pi}\sin^3(u) + C \\ &= \frac{1}{\pi}\sin(\pi x - 1) - \frac{1}{3\pi}\sin^3(\pi x - 1) + C. \end{aligned}$$

13. Using the Pythagorean identity $\sec^2(\theta) = 1 + \tan^2(\theta)$, we get

$$\begin{aligned} \int \sec^4\left(\frac{x}{2}\right)\,dx &= \int \left[1 + \tan^2\left(\frac{x}{2}\right)\right]\sec^2\left(\frac{x}{2}\right)\,dx \\ &= \int \sec^2\left(\frac{x}{2}\right)\,dx + \int \tan^2\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)\,dx \end{aligned}$$

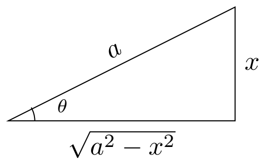
We make the substitution $u = \tan(\frac{x}{2})$, $du = \frac{1}{2}\sec^2(\frac{x}{2})\,dx$ leading us to

$$\begin{aligned} \int \sec^2\left(\frac{x}{2}\right)\,dx + \int \tan^2\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)\,dx &= 2u + \frac{2}{3}u^3 + C \\ &= 2\tan\left(\frac{x}{2}\right) + \frac{2}{3}\tan^3\left(\frac{x}{2}\right) + C. \end{aligned}$$

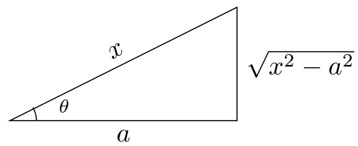
14. incomplete - use $1 - \sin^2(\theta) = \cos^2(\theta)$ and choose $u = \cos(\theta)$

15. incomplete - use $\sec^2(\theta) = \tan^2(\theta) + 1$ and choose $u = \tan(\theta)$

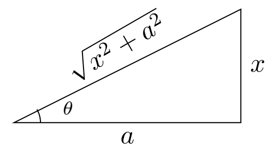
16. $x = a\sin(\theta)$



17. $x = a\sec(\theta)$

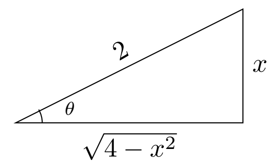


18. $x = a\tan(\theta)$



19. We make the substitution $x = 2\sin(\theta)$, $dx = 2\cos(\theta)\,d\theta$. Then

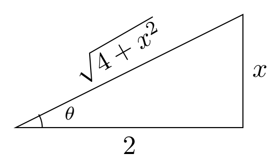
$$\begin{aligned}
\int \frac{-12}{x^2\sqrt{4-x^2}}\,dx &= -12\int \frac{2\cos(\theta)}{4\sin^2(\theta)\sqrt{4-4\sin^2(\theta)}}\,d\theta \\
&= -12\int \frac{2\cos(\theta)}{4\sin^2(\theta)\sqrt{4(\cos^2(\theta))}}\,d\theta \\
&= -12\int \frac{2\cos(\theta)}{4\sin^2(\theta)2\cos(\theta)}\,d\theta \\
&= -3\int \frac{\cos(\theta)}{\sin^2(\theta)\cos(\theta)}\,d\theta \\
&= -3\int \csc^2(\theta)\,d\theta \\
&= 3\cot(\theta) + C \\
&= 3\frac{\sqrt{4-x^2}}{x} + C.
\end{aligned}$$



Reference Triangle, #19.

20. We make the substitution $x = 2\tan(\theta)$, $dx = 2\sec^2(\theta)\,d\theta$. Then

$$\begin{aligned}
\int \frac{x^3}{\sqrt{4+x^2}}\,dx &= \int \frac{8\tan^3(\theta)2\sec^2(\theta)}{\sqrt{4+4\tan^2(\theta)}}\,d\theta \\
&= \int \frac{16\tan^3(\theta)\sec^2(\theta)}{\sqrt{4\sec^2(\theta)}}\,d\theta \\
&= \int \frac{16\tan^3(\theta)\sec^2(\theta)}{2\sec(\theta)}\,d\theta \\
&= 8\int \tan^3(\theta)\sec(\theta)\,d\theta.
\end{aligned}$$

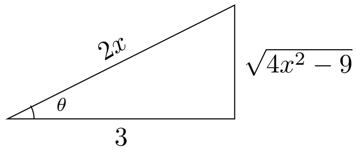


Reference Triangle, #20.

Now, we use a Pythagorean identity and the substitution $u = \sec(\theta)$, $du = \sec(\theta)\tan(\theta)\,d\theta$ to get

$$\begin{aligned}
8\int \tan^3(\theta)\sec(\theta)\,d\theta &= 8\int (\sec^2 - 1)\sec(\theta)\tan(\theta)\,d\theta \\
&= 8\int (u^2-1)\,du \\
&= \frac{8}{3}u^3 - 8u + C \\
&= \frac{8}{3}\sec^3(\theta) - 8\sec(\theta) + C \\
&= \frac{8}{3}\left(\frac{\sqrt{4+x^2}}{2}\right)^3 - 8\frac{\sqrt{4+x^2}}{2} + C \\
&= \frac{\left(\sqrt{4+x^2}\right)^3}{3} - 4\sqrt{4+x^2} + C.
\end{aligned}$$

21. incomplete - choose $2x = 3\sec(\theta)$



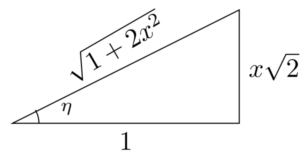
Reference Triangle, #19.

22. *(This can also be done quickly with the Pythagorean identity.)* incomplete - choose $\sqrt{2}\cos(\theta) = \tan(\eta)$
Let $x = \cos(\theta)$, $dx = -\sin(\theta)\,d\theta$. The our integral becomes

$$\int_{\theta=0}^{\theta=\pi/2} \frac{\sin(\theta)}{1+2\cos^2(\theta)}\,d\theta = -\int_{x=1}^{x=0} \frac{dx}{1+2x^2} = \int_{x=0}^{x=1} \frac{dx}{1+2x^2}.$$

Now make the substitution $\sqrt{2}x = \tan(\eta)$, $\sqrt{2}\,dx = \sec^2(\eta)\,d\eta$. Then

$$\begin{aligned}\int_{x=0}^{x=1} \frac{dx}{1+2x^2} &= \int_{x=0}^{x=1} \frac{\frac{1}{\sqrt{2}}\sec^2(\eta)}{1+\tan^2(\eta)}\,d\eta \\ &= \int_{x=0}^{x=1} \frac{\frac{1}{\sqrt{2}}\sec^2(\eta)}{\sec^2(\eta)}\,d\eta\end{aligned}$$



Reference Triangle, #21.

23.

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -3 & 1 & 1 & -2 & 2 \\ 0 & 0 & 0 & -12 & 1 & 3 \\ 1 & 1 & 1 & -1 & 0 & -9 \end{array}\right)$$

24.

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & \frac{423}{2} \\ 0 & 1 & 0 & 0 & 0 & -\frac{407}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 17 \\ 0 & 0 & 0 & 0 & 1 & 207 \end{array}\right)$$

25. The integrand decomposes as:

$$\begin{aligned}\frac{x^2-1}{x^3+x} &= \frac{x^2-1}{x(x^2+1)} \\ &= \frac{A}{x} + \frac{Bx+C}{x^2+1}\end{aligned}$$

incomplete

26. The integrand decomposes as:

$$\begin{aligned}\frac{x^2+x+3}{x^4+6x^2+9} &= \frac{x^2+x+3}{(x^2+3)^2} \\ &= \frac{Ax+B}{(x^2+3)} + \frac{Cx+D}{(x^2+3)^2}.\end{aligned}$$

incomplete

27. The integrand decomposes as:

$$\frac{x-1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x+1)}.$$

So $x-1 = Ax(x+1)+B(x+1)+Cx^2 = (A+C)x^2+(A+B)x+B$, and we can solve to get $A=2$, $B=-1$, $C=-2$. So

$$\int \frac{x-1}{x^2(x+1)}\,dx = \int \frac{2}{x}\,dx + \int \frac{-1}{x^2}\,dx + \int \frac{-2}{(x+1)}\,dx.$$

incomplete

28. The integrand decomposes as:

$$\frac{2x^3+2x^2-4}{(x^2-x)(x^2+4)} = \frac{2(x-1)(x^2+2x+2)}{x(x-1)(x^2+4)} = \frac{2(x^2+2x+2)}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}.$$

So $2x^2+4x+4 = A(x^2+4)+Bx^2+Cx = (A+B)x^2+Cx+4A$, and we can solve to get $A=1$, $B=1$, $C=4$. So

$$\int \frac{2x^3+2x^2-4}{(x^2-x)(x^2+4)}\,dx = \int \frac{1}{x}\,dx + \int \frac{x+4}{x^2+4}\,dx.$$

incomplete

29.

$$\int_1^\infty \frac{dx}{x} = \lim_{t\to\infty} \int_1^t \frac{dx}{x}.$$

incomplete

30.

$$\int_0^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{x^2+1}\,dx$$

incomplete

31. We have to split this into two separate integrals and then take our limits.

$$\begin{aligned}\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}}\,dx &= \int_{-\infty}^0 \frac{e^x}{1+e^{2x}}\,dx + \int_0^\infty \frac{e^x}{1+e^{2x}}\,dx \\ &= \lim_{s\to-\infty} \int_s^0 \frac{e^x}{1+e^{2x}}\,dx + \lim_{t\to\infty} \int_0^t \frac{e^x}{1+e^{2x}}\,dx.\end{aligned}$$

incomplete

32. Our integrand is undefined at $x=0$, so we have to split this into two separate integrals and take our limits from each side.

$$\begin{aligned}\int_{-1}^1 \frac{1}{x^2}\,dx &= \int_{-1}^0 \frac{1}{x^2}\,dx + \int_0^1 \frac{1}{x^2}\,dx \\ &= \lim_{s\to-1^+} \int_s^0 \frac{1}{x^2}\,dx + \lim_{t\to1^-} \int_0^t \frac{1}{x^2}\,dx\end{aligned}$$

incomplete