Recitation 08: Inverse Trig Functions & Applications of Calculus

Joseph Wells
Arizona State University

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Derivatives of inverse trig functions:

$$\frac{d}{dx}[\arcsin(u)] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\operatorname{arccsc}(u)] = -\frac{u'}{|u|\sqrt{u^2-1}}$$
$$\frac{d}{dx}[\arccos(u)] = -\frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\operatorname{arcsec}(u)] = \frac{u'}{\sqrt{u^2-1}}$$
$$\frac{d}{dx}[\arctan(u)] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\operatorname{arccot}(u)] = -\frac{u'}{1+u^2}$$

Here, u is representative of a function as it makes it look a little cleaner and makes the chain rule look a little more obvious. I believe your book gives the formulae in terms of x, so $u = x$ and $u' = 1$.

Example (§3.9, #5). Let $f(x) = (x-1)\arcsin(x)$ on the interval $[-1, 1]$.

a. Graph f with a graphing utility.

b. Compute and graph f'.

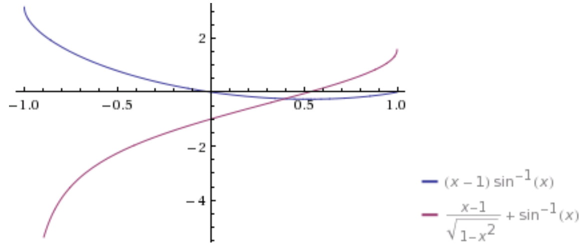
c. Verify that the zeros of f' correspond to points at which f has a horizontal tangent line.

Solution.

a. & b. Via the product rule, we get that $f'(x) = \arcsin(x) + \dfrac{x-1}{\sqrt{1-x^2}}$, the plots of which can be found below. In Maple, we would plot it with the following command:

```
plots[multiple](plot, [(x-1)*arcsin(x), x=-1..1],
                      [arcsin(x)+(x-1)/(sqrt(1-x^2)), x=-1..1] );
```

resulting in a picture like the following:



c. We notice that our derivative crosses the x-axis at $x \approx 0.5$, which is right about the point in our original function where the slope is zero.

Finally, applications of calculus.

Since the derivative tells you the slope of a function at any particular point, it makes sense that we apply it to rates of change. One of the common examples is as follows. Let $f(t)$ (also commonly written as $r(t)$) represent the position of an object as a function of time. We then have the following relationships:

$$\begin{aligned} \text{position function: } & f(t) \\ \text{velocity function: } & \frac{d}{dt}[f(t)] = f'(t) \\ \text{acceleration function: } & \frac{d^2}{dt^2}[f(t)] = f''(t) \end{aligned}$$

Although none of the examples below will explicitly calculate these values, it's useful to know them as they pop up EVERYWHERE.

Example (§3.5, #1). The following figure shows the position function $s = f(t)$ of an airliner on an out-and-back trip from from Seattle to Minneapolis, where $s = f(t)$ is the number of ground miles from Seattle t hours after take-off at 6:00 AM. The plane returns to Seattle 8.5 hours later at 2:30 PM.

a. Calculate the average velocity of the airliner during the first 1.5 hours of the trip $(0 \leq t \leq 1.5)$.

b. Calculate the average velocity of the airliner between 1:30 PM and 2:30 PM $(7.5 \leq t \leq 8.5)$.

c. At what time(s) is the velocity 0? Give a plausible explanation.

d. Determine the velocity of the airliner at noon $(t = 6)$ and explain why the velocity is negative.

Solution

a. The average velocity is

$$v_{\text{avg}} = \frac{f(t_1) - f(t_0)}{t_1 - t0} = \frac{600 \text{ mi} - 0 \text{ mi}}{1.5 \text{ h} - 0 \text{ h}} = 450 \text{ mi/h}.$$

b. The average velocity is

$$v_{\text{avg}} = \frac{f(t_1) - f(t_0)}{t_1 - t0} = \frac{0 \text{ mi} - 350 \text{ mi}}{8.5 \text{ h} - 7.5 \text{ h}} = -350 \text{ mi/h}.$$

c. The velocity (slope) is 0 when $3 \leq t \leq 5$. This would represent the time that the plane landed in Minneapolis and partook in all of its plane-ly business before taking off again.

d. The tangent line at $t = 6$ looks to be about the same as the portion of the graph where $6 \leq t \leq 7$, so $f'(6) \approx -600$ mi/h. The reason it is negative is because it represents the plane getting closer to Seattle.

Example (§3.5, #3)**.** The graph shows the position $s = f(t)$ of a car t hours after 5:00 PM relative to its starting point $s = 0$, where s is measured in miles.

a. Describe the velocity of the car. Specifically, when is it speeding up and when is it slowing down?

b. At approximately what time is the car traveling the fastest? The slowest?

c. What is the approximate maximum velocity of the car? The approximate minimum velocity?

solution

a. The car is speeding up when our slope increases and slowing down when the slope decreases. Looking at the graph, the car appears to speed up at 1.75 hours and slows down at approximately 1.25 hours.

b. The maximum velocity corresponds to the steepest slope, which occurs in the time interval $[0.25, 0.5]$. The minimum velocity corresponds to the least-steep slope, which occurs in the time interval $[1.25, 1.5]$.

c. From part b., we get that the maximum velocity is approximately $\dfrac{20\text{mi} - 0.75\text{mi}}{0.5\text{h} - 0.25\text{h}} =$

77mi/h and the minimum velocity is approximately $\dfrac{41\text{mi} - 40\text{mi}}{1.5\text{h} - 1.25\text{h}} = 4\text{mi/h}$.

Example (§3.10, #1)**.** The volume of a cube decreases at a rate of 0.5 ft^3/min. What is the rate of change of the side length when the side lengths are 12 ft?

Solution

Recall that the volume of the cube is given by $V = s^3$, where s is the length of the sides. Ultimately, the volume and the side lengths are changing with respect to time, so we can differentiate both sides with respect to time t and get

$$\begin{aligned} \frac{d}{dt}[V] &= \frac{d}{dt}[s^3] \\ &= 3s^2 \frac{ds}{dt} \end{aligned}$$

In particular, we're given that $\frac{dV}{dt} = -0.5$ ft^3/min when $s = 12$ ft, so we can plug them into our equation and find $\frac{ds}{dt}$, the rate of change of the side lengths.

$$\begin{aligned}
3s^2\frac{ds}{dt} &= \frac{dV}{dt} \\
\frac{ds}{dt} &= \frac{1}{3s^2}\frac{dV}{dt} \\
&= \frac{1}{3(12\text{ ft})^2}\cdot\frac{-1\text{ ft}^3}{2\text{ min}} \\
&= -\frac{1}{864}\text{ ft/min.}
\end{aligned}$$

Assignment

MAT270 Recitation Notebook
§3.5, Problems 2
§3.9, Problems 1,2
§3.10, Problems 1,5