MAT266 Homework 09 (Solutions)

1. **a.** We have that the Taylor series for $\cos(x)$ at $a = 0$ is

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

whence

$$\cos(x^3) = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}.}$$

b. It's not too hard to see that

$$f^{(n)}(x) = 2^n e^{2x} \quad \Rightarrow \quad f^{(n)}(3) = 2^n e^6$$

whence the Taylor expansion around $a = 3$ is

$$e^{2x} = \boxed{\sum_{n=0}^{\infty} \frac{2^n e^6 (x-3)^n}{n!}}$$

c. It's not too hard to see that

$$f^{(n)}(x) = \frac{(-1)^n \cdot n!}{x^{n+1}} \quad \Rightarrow \quad f^(n)(-3) = \frac{(-1)^n \cdot n!}{(-3)^{n+1}} = -\frac{n!}{3^{n+1}}$$

whence the Taylor expansion around $a = -3$ is

$$\frac{1}{x} = \boxed{\sum_{n=0}^{\infty} -\frac{1}{3^{n+1}}(x+3)^n.}$$

d. We have that the Taylor series for $\cos(x)$ at $a = 0$ is

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

whence the Taylor series expansion of $\sin x$ around $a = \frac{\pi}{2}$ is

$$\sin x = \cos\left(x - \frac{\pi}{2}\right) = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(x - \frac{\pi}{2}\right)^{2n}.}$$

2. Knowing that

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^n + 1x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

we get

$$\begin{aligned}\frac{x - \ln(1+x)}{x^2} &= \frac{x - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)}{x^2} \\ &= \frac{\frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} + \cdots}{x^2} \\ &= \frac{1}{2} - \frac{x}{3} + \frac{x^2}{4} - \cdots\end{aligned}$$

And thus

$$\lim_{x\to 0} \frac{x - \ln(1+x)}{x^2} = \lim_{x \to 0} \left(\frac{1}{2} - \frac{x}{3} + \frac{x^2}{4} - \cdots\right) = \boxed{\frac{1}{2}.}$$

3. For the most accurate approximation, we should center our series around 0. Using the (Maclaurin) binomial series with $k = -1/2$, we get

$$\begin{aligned}\frac{1}{\sqrt{1+x^3}} &= \left(1 + x^3\right)^{-1/2} \\ &= 1 + \frac{\left(-\frac{1}{2}\right)}{1!}x^3 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}x^6 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}x^9 + \cdots\end{aligned}$$

Taking the 9^{th} order Maclaurin polynomial (it may be overkill, but it's only a few nonzero terms), we approximate that

$$\frac{1}{\sqrt{1+x^3}} \approx 1 - \frac{x^3}{2} + \frac{3x^6}{8} - \frac{5x^9}{16}.$$

Thus

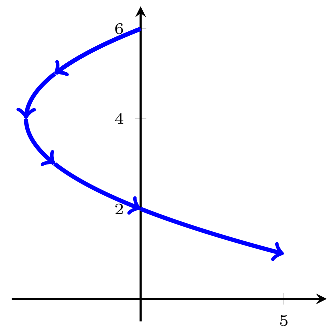
$$\begin{aligned}\int_0^{0.1} \frac{dx}{\sqrt{1+x^3}} &\approx \int_0^{0.1} 1 - \frac{x^3}{2} + \frac{3x^6}{8} - \frac{5x^9}{16}\, dx \\ &= \boxed{0.0999875.}\end{aligned}$$

In fact, according to our favorite computer algebra system

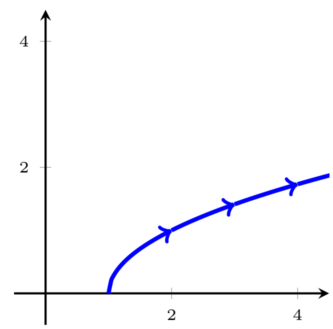
$$\int_0^{0.1} \frac{dx}{\sqrt{1+x^3}} \approx 0.0999875$$

and so our approximation is accurate to *at least* 5 decimal places.

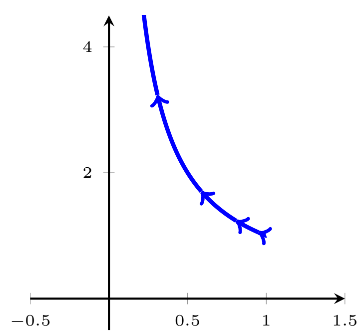
4. The Cartesian equation of this curve is $\boxed{x = (y-4)^2 - 4, \quad 1 \le y \le 5.}$



5. The Cartesian equation of this curve is $\boxed{y = \sqrt{x-1}.}$



6. The Cartesian equation of this curve is $\boxed{y = \frac{1}{x}, \quad 0 \le x < 1.}$



7. The Cartesian equation of this curve is $\boxed{\frac{x^2}{2^2} + \frac{(y-1)^2}{1^2} = 1.}$