MAT266 Homework 07 (Solutions)

1. $a_n = \frac{(-1)^{n+1}n^2}{n+1}$

2. **a.** Since $|0.2| < 1$,

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} [1-(0.2)^n] \\ &= \left[\lim_{n\to\infty} 1\right] - \left[\lim_{n\to\infty} (0.2)^n\right] \\ &= 1 - 0 = \boxed{1,\ \text{convergent.}}\end{aligned}$$

b. Since $\left|\frac{3}{5}\right| < 1$,

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{3^n}{5^{n+2}} \\ &= \lim_{n\to\infty} \frac{1}{25}\left(\frac{3}{5}\right)^n \\ &= \frac{1}{25}\cdot\left[\lim_{n\to\infty}\left(\frac{3}{5}\right)^n\right] \\ &= \frac{1}{25}\cdot 0 = \boxed{0,\ \text{convergent.}}\end{aligned}$$

c. Note that $-\frac{1}{2\sqrt{n}} < a_n < \frac{1}{2\sqrt{n}}$ for each n. We have

$$\begin{aligned}\lim_{n\to\infty} -\frac{1}{2\sqrt{n}} &= 0, \\ \lim_{n\to\infty} \frac{1}{2\sqrt{n}} &= 0.\end{aligned}$$

Thus, by the squeeze theorem,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n}{2\sqrt{n}} = \boxed{0,\ \text{convergent.}}$$

3. **a.** We see that this is a geometric series with ratio $r = 0.25$ and first term $a = 2$. Since $|r| < 1$,

$$2 + 0.5 + 0.125 + 0.03125 + \cdots = \frac{a}{1-r} = \frac{2}{1-0.25} = \boxed{\frac{8}{3},\ \text{convergent.}}$$

b. We see that this is a geometric series with ratio $r = -\frac{10}{9}$ and first term $a = 10$. Since $|r| \geq 1$, the series $\boxed{\text{diverges}}$.

4. Determine whether the series is convergent or divergent. If it is convergent, find its sum.

a. Since

$$\lim_{k\to\infty} \frac{k(k+2)}{(k+3)^2} = 1 \neq 0$$

the series $\boxed{\text{diverges}}$ by the Divergence Test.

b. We have that $\sum_{n=1}^{\infty} \frac{1}{3^n}$ is a geometric series with ratio $r = \frac{1}{3}$ and first term $a = \frac{1}{3}$, and $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$ is a geometric series with ratio $r = \frac{2}{3}$ and first term $a = \frac{2}{3}$. Each of these geometric series converges by the geometric series test, and thus

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1+2^n}{3^n} = \sum_{n=1}^{\infty}\left(\frac{1}{3^n} + \frac{2^n}{3^n}\right) &= \sum_{n=1}^{\infty}\frac{1}{3^n} + \sum_{n=1}^{\infty} \\ &= \left(\frac{\frac{1}{3}}{1-\frac{1}{3}}\right) + \left(\frac{\frac{2}{3}}{1-\frac{2}{3}}\right) \\ &= \frac{1}{2} + 2 = \boxed{\frac{5}{2}, \text{ convergent.}}\end{aligned}$$

c. Since the limit

$$\lim_{k\to\infty} (\cos 1)^k$$

does not exist (and in particular, is not 0). So, by the Divergence Test, the series is $\boxed{\text{divergent.}}$

d. Recall that $\ln\left(\frac{n}{n+1}\right) = \ln(n) - \ln(n+1)$. Letting $s_k = \sum_{n=1}^{k} a_n$, we have

$$\begin{aligned}s_1 &= \ln 1 - \ln 2 = -\ln 2 \\ s_2 &= \ln 1 - \ln 2 + \ln 2 - \ln 3 = -\ln 3 \\ s_3 &= \ln 1 - \ln 2 + \ln 2 - \ln 3 + \ln 3 - \ln 4 = 1 - \ln 4 \\ &\vdots \\ s_k &= -\ln k.\end{aligned}$$

And since $s_k \to -\infty$ as $k \to \infty$, the series $\boxed{\text{diverges.}}$.

5. a. We have that $a_n = \frac{n}{5^n}$ and $a_{n+1} = \frac{n+1}{5^{n+1}}$. Thus

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{n+1}{5^{n+1}} \cdot \frac{5^n}{n}\right| \\ &= \lim_{n\to\infty}\left|\frac{1}{5}\cdot\frac{n+1}{n}\right| = \frac{1}{5}.\end{aligned}$$

By the ratio test, this series is $\boxed{\text{absolutely convergent.}}$

b. We have that $a_n = \frac{(-10)^n}{n!}$ and $a_{n+1} = \frac{(-10)^{n+1}}{(n+1)!}$. Thus

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-10)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-10)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{-10}{n+1}\right| = 0.\end{aligned}$$

By the ratio test, this series is $\boxed{\text{absolutely convergent.}}$

c. By the ratio test, this series is $\boxed{\text{absolutely convergent.}}$

d. By the ratio test, this series is $\boxed{\text{absolutely convergent.}}$