MAT266 Homework 06 (Solutions)

1. We have that $y' = -\frac{x}{\sqrt{2-x^2}}$, and so the radicand for our arc length integral is

$$\begin{aligned} 1 + (y')^2 &= 1 + \frac{x^2}{2-x^2} \\ &= \frac{2-x^2}{2-x^2} + \frac{x^2}{2-x^2} \\ &= \frac{2}{2-x^2} \end{aligned}$$

Thus the length of the curve on the given interval is

$$\begin{aligned} L = \int_0^1 \sqrt{\frac{2}{2-x^2}}\,dx &= \sqrt{2}\int_0^1 \frac{1}{\sqrt{2-x^2}}\,dx \\ &= \sqrt{2}\left[\arcsin\left(\frac{x}{\sqrt{2}}\right)\right]_0^1 \\ &= \boxed{\sqrt{2}\cdot\frac{\pi}{4} \approx 1.11072} \end{aligned}$$

2. We have that $y' = 9x^{1/2}$, and so the radicand for our arc length integral is

$$1 + (y')^2 = 1 + 81x$$

Thus the length of the curve on the given interval is

$$\begin{aligned} L &= \int_0^1 \sqrt{1+81x}\,dx \\ &= \left[\frac{2}{243}(1+81x)^{3/2}\right]_0^1 \\ &= \boxed{\frac{2\cdot 82^{3/2}}{243} - \frac{2}{243} \approx 6.13022} \end{aligned}$$

3. We have that $y' = \tan x$, and so the radicand for our arc length intergral is

$$1 + (y')^2 = 1 + \tan^2 x = \sec^2 x.$$

Thus the arc length of the curve on the given interval is

$$\begin{aligned} L &= \int_0^{\pi/4} \sqrt{\sec^2 x}\,dx \\ &= \int_0^{\pi/4} \sec x\,dx \\ &= [\ln|\sec x + \tan x|]_0^{\pi/4} \\ &= \boxed{\ln(1+\sqrt{2}) \approx 0.88137} \end{aligned}$$

4. We have that $y' = \frac{x}{2} - \frac{1}{2x}$, so the radicand for our arc length integral is

$$\begin{aligned} 1 + (y')^2 &= 1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2 \\ &= 1 + \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2} \\ &= \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2} \\ &= \left(\frac{x}{2} + \frac{1}{2x}\right)^2 \end{aligned}$$

Thus the length of the curve on the given interval is

$$\begin{aligned} L &= \int_1^2 \sqrt{\left(\frac{x}{2} + \frac{1}{2x}\right)^2}\, dx \\ &= \int_1^2 \frac{x}{2} + \frac{1}{2x}\, dx \\ &= \left[\frac{1}{4}x^2 - \frac{1}{2}\ln x\right]_1^2 \\ &= \boxed{\frac{3}{4} + \frac{\ln 2}{2} \approx 1.09657} \end{aligned}$$

5. Note: we have to convert all lengths given into feet. To compute the spring constant, we use Hooke's law:

$$\begin{aligned} f(x) &= kx \\ 10\,\text{lb} &= k(4\,\text{in}) \\ 10\,\text{lb} &= k(\frac{1}{3}\,\text{ft}) \\ \Rightarrow \quad k &= 30\,\text{lb/ft}. \end{aligned}$$

Thus the amount of work required to stretch the string from its natural length to $6\,\text{in} = \frac{1}{2}\,\text{ft}$ beyond its natural length is

$$\begin{aligned} W = \int_0^{1/2} f(x)\, dx &= \int_0^{1/2} 30x\, dx \\ &= \left[15x^2\right]_0^{1/2} \\ &= \boxed{\frac{15}{4}\ \text{ft-lb}} \end{aligned}$$

6. We assume that the chain has uniform density, so each meter of chain has a mass of 8 kg. When y meters of the chain is lifted from the ground, the force of gravity is given by

$$F(y) = mg = (9.8)8y = 78.4y$$

The amount of work required to raise one end of the chain to 6 m is thus

$$W = \int_0^6 78.4y\,dy = \left[39.2y^2\right]_0^6 = 1411.2\,\text{ft-lb}$$

7. We place the bottom of the tank at the origin. A vertical cross-sectional slice of the paraboloid is the parabola $y = x^2/4$, and so at each height y, the slice of water is a circle of radius $\sqrt{4y}$. Thus, the volume of each infinitesimal slice is $V(y) = 4\pi y\,dy$.

a. The work done in emptying the tank is

$$\begin{aligned} W &= \int_0^4 \rho g(4-y)V(y) \\ &= 4\pi(62.5)\int_0^4 4y - y^2\,dy \\ &= 250\pi\left[2y^2 - \frac{1}{3}y^3\right]_0^4 \\ &= \boxed{\frac{8000\pi}{3} \approx 8377.6\,\text{ft-lb}} \end{aligned}$$

b. After 4000 ft-lb of work has been done, h feet of water have been drained. This is

$$\begin{aligned} 4000 &= 250\pi\int_0^h 4y - y^2\,dy \\ &= -\frac{250}{3}\pi h^2(h-6) \end{aligned}$$

Solving for h, we get that $h \approx 1.93989$ ft, so there is $4 - h \approx \boxed{2.06011\,\text{ft}}$ of water left in the tank.