MAT266 Homework 04 (Solutions)

1. This is a Type I improper integral, and we write

$$\int_0^\infty \sqrt[4]{1+x}\,dx = \lim_{b\to\infty}\int_0^b \sqrt[4]{1+x}\,dx.$$

We make a substitution

$$u = 1+x$$
$$du = dx$$

and our limits of integration become

$$u(0) = 0+1 = 1$$
$$u(b) = b+1$$

Since $\lim_{b\to\infty} b+1 = \infty$, we can simply write the upper limit of integration as b without affecting the limit. Thus we get

$$\begin{aligned}\lim_{b\to\infty}\int_0^b \sqrt[4]{1+x}\,dx &= \lim_{b\to\infty}\int_1^{b+1} u^{1/4}\,du \\ &= \lim_{b\to\infty}\left[\frac{4}{5}u^{5/4}\right]_1^{b+1} \\ &= \lim_{b\to\infty}\left[\frac{4}{5}b^{5/4} - \frac{4}{5}\right].\end{aligned}$$

The limit above does not exist, and so the integral is divergent.

2. This is a Type I improper integral, and we write

$$\begin{aligned}\int_{-\infty}^\infty \cos(\pi t)\,dt &= \lim_{a\to-\infty}\int_a^0 \cos(\pi t)\,dt + \lim_{b\to\infty}\int_0^b \cos(\pi t)\,dt \\ &= \lim_{a\to-\infty}\left[\frac{1}{\pi}\sin(\pi t)\right]_a^0 + \lim_{b\to\infty}\left[\frac{1}{\pi}\sin(\pi t)\right]_0^b \\ &= \lim_{a\to-\infty} -\frac{1}{\pi}\sin(\pi a) + \lim_{b\to\infty}\frac{1}{\pi}\sin(\pi b).\end{aligned}$$

Since neither of these limits exist, then the integral is divergent.

3. $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx$

This is a Type I improper integral, and we write

$$\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x}\,dx$$

Using the substitution

$$u = \ln x$$
$$du = \frac{1}{x}\,dx$$

our limits of integration are

$$u(1) = \ln(1) = 0$$
$$u(b) = \ln(b)$$

Because $\lim_b \ln(b) \to \infty$, we can simply write the upper limit of integration as b without affecting the limit. Thus we get

$$\begin{aligned}\lim_{b\to\infty} \int_1^b \frac{\ln x}{x}\, dx &= \lim_{b\to\infty} \int_0^b u\, du \\ &= \lim_{b\to\infty} \left[\frac{1}{2}u^2\right]_0^b \\ &= \lim_{b\to\infty} \frac{1}{2}u^2\end{aligned}$$

Since this limit does not exist, the integral is divergent.

4. We can apply the "p-test" (as discussed in class) to see that this integral converges. So, since this is a Type I integral, we write

$$\begin{aligned}\int_1^\infty \frac{3}{x^5}\, dx &= \lim_{b\to\infty} \int_1^b \frac{3}{x^5}\, dx \\ &= \lim_{b\to\infty} \left[-\frac{3}{4x^4}\right]_1^b \\ &= \lim_{b\to\infty} -\frac{3}{b^4} + \frac{3}{4} \\ &= \boxed{\frac{3}{4}}\end{aligned}$$

5. This is a Type II improper integral and the integrand is undefined at $w = 2$, so we write

$$\int_0^5 \frac{w}{w-2}\, dw \lim_{b\to 2^-} \int_0^b \frac{w}{w-2}\, dw + \lim_{a\to 2^+} \int_a^5 \frac{w}{w-2}\, dw$$

To save some headache, we'll solve the indefinite integral by first making a substitution

$$u = w - 2$$
$$du = dw$$

which gets us

$$\int \frac{w}{w-2}\, dw = \int \frac{u+2}{u}\, du = \int 1 + \frac{2}{u}\, du = u + 2\ln|u| + C = w + 2\ln|w-2| + C$$

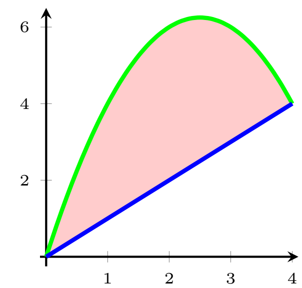
Thus

$$\begin{aligned}&\lim_{b\to 2^-} \int_0^b \frac{w}{w-2}\, dw + \lim_{a\to 2^+} \int_a^5 \frac{w}{w-2}\, dw \\ &= \lim_{b\to 2^-} [w + 2\ln|w-2|]_0^b + \lim_{a\to 2^+} [w + 2\ln|w-2|]_a^5 \\ &= \lim_{b\to 2^-} (b + 2\ln|b-2| - 2\ln(2)) + \lim_{a\to 2^+} (5 + 2\ln(3) - a + \ln|a-2|)\end{aligned}$$

Since neither of these limits exist, then the integral is divergent.

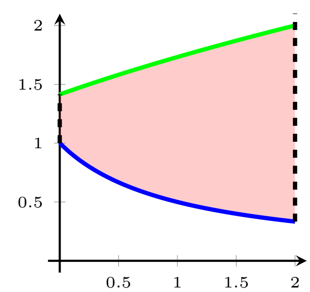
6. After seeing the graph, we choose to partition the x-axis (hence integrating with respect to x). The two curves intersect when $x = 0$ and $x = 4$, and get that the area between the curves is

$$\begin{aligned} A &= \int_0^4 (5x - x^2) - x\,dx \\ &= \int_0^4 4x - x^2\,dx \\ &= \left[2x^2 - \frac{1}{3}x^3\right]_0^4 \\ &= \boxed{\frac{32}{3}} \end{aligned}$$



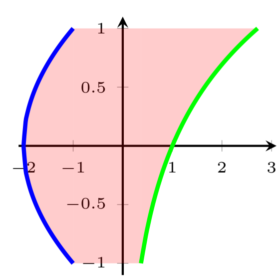
7. After seeing the graph, we choose to partition the x-axis (hence integrating with respect to x) and get that the area of the region is

$$\begin{aligned} A &= \int_0^2 \sqrt{x+2} - \frac{1}{x+1}\,dx \\ &= \int_0^2 \sqrt{x+2}\,dx - \int \frac{1}{x+1}\,dx \\ &= \left[\frac{2}{3}(x+2)^{3/2}\right]_0^2 - [\ln|x+1|]_0^2 \\ &= \boxed{\frac{16}{3} - \frac{4\sqrt{2}}{3} - \ln(3) \approx 2.3491} \end{aligned}$$



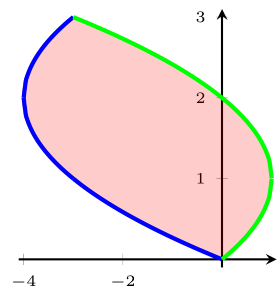
8. After seeing the graph, we choose to partition the y-axis (hence integrating with respect to y) and get that the area of the region is

$$\begin{aligned} A &= \int_{-1}^1 e^y - (y^2 - 2)\,dy \\ &= \int_{-1}^1 e^y - y^2 + 2\,dy \\ &= \left[e^y - \frac{1}{3}y^3 + 2y\right]_{-1}^1 \\ &= \boxed{\frac{10}{3} - e^{-1} + e \approx 5.6837} \end{aligned}$$



9. After looking at the graph, we choose to partition the y-axis (hence integrating with respect to y). The curves intersect when $y = 0$ and $y = 3$, so the area of the region is. $x = y^2 - 4y$, $x = 2y - y^2$

$$\begin{aligned} A &= \int_0^3 (2y - y^2) - (y^2 - 4y)\, dy \\ &= \int_0^3 6y - 2y^2\, dy \\ &= \left[3y - \frac{2}{3}y^3\right]_0^3 \\ &= \boxed{9} \end{aligned}$$



10. $y = \cos x$, $y = \sin(2x)$, $x = 0$, $x = \frac{\pi}{2}$. Note that this region consists of two parts.

$$\begin{aligned} A &= \int_0^{\pi/6} \cos x - \sin(2x)\, dx + \int_{\pi/6}^{\pi/2} \sin(2x) - \cos x\, dx \\ &= \left[\sin x + \frac{1}{2}\cos(2x)\right]_0^{\pi/6} + \left[-\frac{1}{2}\cos(2x) - \sin x\right]_{\pi/6}^{\pi/2} \\ &= \boxed{\frac{1}{2}} \end{aligned}$$