MAT266 Homework 02 (Solutions)

1. **a.** Using the Pythagorean identity, we can rewrite this integral as

$$\int \sin^3\theta \cos^4\theta \, d\theta = \int \sin\theta(1-\cos^2\theta)\cos^4\theta \, d\theta$$

Now choosing the substitution

$$\begin{aligned} u &= \cos\theta \\ du &= -\sin\theta \, d\theta \end{aligned}$$

we get

$$\begin{aligned} \int \sin\theta(1-\cos^2\theta)\cos^4\theta \, d\theta &= -\int (1-u^2)u^4 \, du \\ &= -\int u^4 - u^6 \, du \\ &= -\left(\frac{1}{5}u^5 - \frac{1}{7}u^7\right) + C \\ &= \boxed{-\frac{1}{5}\cos^5\theta + \frac{1}{7}\cos^7\theta + C} \end{aligned}$$

b. We can apply a power-reducing formula to rewrite this integral as

$$\begin{aligned} \int_0^\pi \cos^4(2t)\, dt &= \int_0^\pi \left(\cos^2(2t)\right)^2 \, dt \\ &= \int_0^\pi \left(\frac{1}{2} + \frac{1}{2}\cos(4t)\right)^2 \, dt \\ &= \int_0^\pi \frac{1}{4} + \cos(4t) + \frac{1}{2}\cos^2(4t)\, dt. \end{aligned}$$

Applying the power-reducing formula again gives us

$$\begin{aligned} \int_0^\pi \frac{1}{4} + \cos(4t) + \frac{1}{2}\cos^2(4t)\, dt &= \int_0^\pi \frac{1}{4} + \cos(4t) + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\cos(8t)\right) \, dt \\ &= \int_0^\pi \frac{1}{4} + \cos(4t) + \frac{1}{4} + \frac{1}{4}\cos(8t)\, dt \\ &= \left[\frac{1}{4}t + \frac{1}{4}\sin(4t) + \frac{1}{4}t + \frac{1}{32}\sin(8t)\right]_0^\pi \\ &= \boxed{\frac{\pi}{2}} \end{aligned}$$

c. We use the substitution

$$\begin{aligned} u &= \sec(x) \\ du &= \sec(x)\tan(x)\, dx \end{aligned}$$

to get

$$\begin{aligned}\int \tan(x)\sec^3(x)\,dx &= \int u^2\,du \\ &= \frac{1}{3}u^3 + C \\ &= \boxed{\frac{1}{3}\sec^3(x) + C}\end{aligned}$$

d. We apply a Pythagorean identity to rewrite the integral as

$$\begin{aligned}\int \tan^2(x) + \tan^4(x)\,dx &= \int \tan^2(x)\left(1 + \tan^2(x)\right)\,dx \\ &= \int \tan^2(x)\sec^2(x)\,dx.\end{aligned}$$

Using the substitution

$$\begin{aligned}u &= \tan(x) \\ du &= \sec^2(x)\,dx\end{aligned}$$

we get

$$\begin{aligned}\int \tan^2(x)\sec^2(x)\,dx &= \int u^2\,du \\ &= \frac{1}{3}u^3 + C \\ &= \boxed{\frac{1}{3}\tan^3(x) + C}\end{aligned}$$

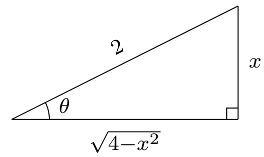
2. **a.** Making the substitution

$$\begin{aligned}x &= 2\sin\theta \\ dx &= 2\cos\theta\,d\theta\end{aligned}$$

we get

$$\begin{aligned}\int \frac{1}{x^2\sqrt{4-x^2}}\,dx &= \int \frac{1}{4\sin^2\theta\sqrt{4-4\sin^2\theta}}\cdot 2\cos\theta\,d\theta \\ &= \int \frac{2\cos\theta}{8\sin^2\theta\cos\theta}\,d\theta \\ &= \int \frac{1}{4\sin^2\theta}\,d\theta \\ &= \frac{1}{4}\int \csc^2\theta\,d\theta \\ &= -\frac{1}{4}\cot\theta + C\end{aligned}$$

Our choice in substitution can be rewritten as $\sin\theta = \frac{x}{2}$, and so we get the triangle



From the triangle, we can deduce that

$$\int \frac{1}{x^2\sqrt{4-x^2}}\,dx = -\frac{1}{4}\cot\theta + C = -\frac{1}{4}\cdot\frac{\sqrt{4-x^2}}{x} + C$$

b. Making the substitution

$$\begin{aligned} x &= 2\tan\theta \\ dx &= 2\sec^2\theta\,d\theta \end{aligned}$$

we get

$$\begin{aligned} \int \frac{x^3}{\sqrt{x^2+4}}\,dx &= \int \frac{8\tan^3\theta}{\sqrt{4\tan^2\theta+4}}\cdot 2\sec^2\theta\,d\theta \\ &= \int \frac{8\tan^3\theta\sec^2\theta}{\sec\theta}\,d\theta \\ &= 8\int \tan^3\theta\sec\theta\,d\theta \\ &= 8\int (\sec^2\theta - 1)\sec\theta\tan\theta\,d\theta \end{aligned}$$

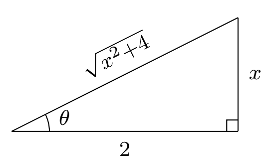
Now make the substitution

$$\begin{aligned} u &= \sec\theta \\ du &= \sec\theta\tan\theta \end{aligned}$$

we get

$$\begin{aligned} 8\int (\sec^2\theta - 1)\sec\theta\tan\theta\,d\theta &= 8\int (u^2-1)\,du \\ &= \frac{8}{3}u^3 - 8u + C \\ &= \frac{8}{3}\sec^3\theta - 8\sec\theta + C. \end{aligned}$$

Our choice in substitution can be rewritten as $\tan\theta = \frac{x}{2}$, and so we get the triangle



From the triangle, we can deduce that

$$\int \frac{x^3}{\sqrt{x^2+4}}\,dx = \frac{8}{3}\sec^3\theta - 8\sec\theta + C = \boxed{\frac{8}{3}\left(\frac{\sqrt{x^2+4}}{2}\right)^3 - 8\frac{\sqrt{x^2+4}}{2} + C}$$

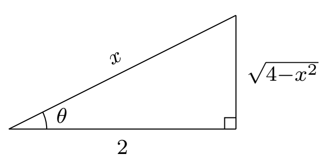
c. Making the substitution

$$\begin{aligned} x &= 2\sec\theta \\ dx &= 2\sec\theta\tan\theta\,d\theta \end{aligned}$$

we get

$$\begin{aligned} \int \frac{\sqrt{x^2-4}}{x}\,dx &= \int \frac{\sqrt{4\sec^2\theta-4}}{2\sec\theta}\cdot 2\sec\theta\tan\theta\,d\theta \\ &= \int \frac{4\sec\theta\tan^2\theta}{2\sec\theta}\,d\theta \\ &= \int 2\tan^2\theta\,d\theta \\ &= \int 2(\sec^2\theta-1)\,d\theta \\ &= 2\tan\theta - 2\theta + C. \end{aligned}$$

Our choice in substitution can be rewritten as $\sec\theta = \frac{x}{2}$, and so we get the reference triangle



From the reference triangle we get

$$\int \frac{\sqrt{x^2-4}}{x}\,dx = 2\tan\theta - 2\theta + C = \boxed{\sqrt{4-x^2} - 2\,\mathrm{arcsec}\left(\frac{x}{2}\right) + C}$$

3. Evaluate the integral.

a. Making the substitution

$$\begin{aligned} t &= \sec\theta \\ dt &= \sec\theta\tan\theta\,d\theta \end{aligned}$$

our new limits of integration become

$$\begin{aligned} \sqrt{2} = \sec(\theta) \quad &\Rightarrow \quad \theta = \frac{\pi}{4} \\ 2 = \sec(\theta) \quad &\Rightarrow \quad \theta = \frac{\pi}{3} \end{aligned}$$

and we get

$$\begin{aligned}
\int_{t=\sqrt{2}}^{t=2} \frac{1}{t^3\sqrt{t^2-1}}\,dt &= \int_{\theta=\pi/4}^{\theta=\pi/3} \frac{1}{\sec^3\theta\sqrt{\sec^2\theta-1}}\cdot\sec\theta\tan\theta\,d\theta \\
&= \int_{\pi/4}^{\pi/3} \frac{1}{\sec^2\theta}\,d\theta \\
&= \int_{\pi/4}^{\pi/3} \cos^2\theta\,d\theta \\
&= \int_{\pi/4}^{\pi/3} \frac{1+\cos(2\theta)}{2}\,d\theta \\
&= \left[\frac{\theta}{2}+\frac{\sin(2\theta)}{4}\right]_{\pi/4}^{\pi/3} \\
&= \boxed{\frac{\sqrt{3}}{8}-\frac{1}{4}+\frac{\pi}{24}\approx 0.09741}
\end{aligned}$$

b. Making the substitution

$$\begin{aligned}
x &= 2\tan\theta \\
dx &= 2\sec^2\theta\,d\theta
\end{aligned}$$

our new limits of integration become

$$\begin{aligned}
0 = 2\tan\theta \quad &\Rightarrow \quad \theta = 0 \\
2 = 2\tan\theta \quad &\Rightarrow \quad \theta = \frac{\pi}{4}
\end{aligned}$$

and we get

$$\begin{aligned}
\int_{x=0}^{x=2} x^3\sqrt{x^2+4}\,dx &= \int_{\theta=0}^{\theta=\pi/4} 8\tan^3\theta\sqrt{4\tan^2\theta+4}\cdot 2\sec^2\theta\,d\theta \\
&= \int_0^{\pi/4} 32\tan^3\theta\sec^3\theta\,d\theta \\
&= 32\int_0^{\pi/4} (\sec^2\theta-1)\sec^2\theta\cdot\sec\theta\tan\theta\,d\theta
\end{aligned}$$

Making another substitution

$$\begin{aligned}
u &= \sec\theta \\
du &= \sec\theta\tan\theta\,d\theta
\end{aligned}$$

our new limits of integration become

$$\begin{aligned}
u(0) &= 1 \\
u(\frac{\pi}{4}) &= \sqrt{2}
\end{aligned}$$

and so we get

$$32\int_{\theta=0}^{\theta=\pi/4}(\sec^2\theta-1)\sec^2\theta\cdot\sec\theta\tan\theta\,d\theta = 32\int_{u=1}^{u=\sqrt{2}}(u^2-1)u^2\,du$$
$$= 32\left[\frac{1}{5}u^5-\frac{1}{3}u^3\right]_1^{\sqrt{2}}$$
$$= \boxed{\frac{64}{15}(1+\sqrt{2})\approx 10.30064}$$

4. Perform the partial fraction decomposition.

a. We solve for the unknown constants A and B in the following equation

$$\frac{1+6x}{(4x-3)(2x+5)} = \frac{A}{4x-3}+\frac{B}{2x+5}$$
$$\Rightarrow \quad 1+6x = A(2x+5)+B(4x-3)$$

When $x=\frac{3}{4}$ we get that $A=\frac{11}{13}$, and when $x=-\frac{5}{2}$ we get that $B=\frac{14}{13}$. So the partial fraction decomposition is

$$\frac{1+6x}{(4x-3)(2x+5)} = \boxed{\frac{11}{13(4x-3)}+\frac{14}{13(2x+5)}}$$

b. After a bit of factoring the denominator, we solve for the unknown constants A, B, and C in the following equation

$$\frac{10}{5x^2-2x^3} = \frac{10}{x^2(5-2x)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{5-2x}$$
$$\Rightarrow \quad 10 = Ax(5-2x)+B(5-2x)+Cx^2$$

When $x=0$ we get that $B=2$, when $x=\frac{5}{2}$ we get that $C=\frac{8}{5}$, and when $x=1$ (combined with our knowledge of B and C) we get that $A=\frac{4}{5}$. So the partial fraction decomposition is

$$\frac{10}{5x^2-2x^3} = \boxed{\frac{4}{5x}+\frac{2}{x^2}+\frac{8}{5(5-2x)}}$$

c. After a bit of factoring the denominator, we solve for the unknown constants A and B in the following equation

$$\frac{x}{x^2+x-2} = \frac{x}{(x-1)(x+2)} = \frac{A}{x-1}+\frac{B}{x+2}$$
$$\Rightarrow \quad x = A(x+2)+B(x-1)$$

When $x=1$ we have that $A=\frac{1}{3}$, and when $x=-2$ we have that $B=\frac{2}{3}$. So the partial fraction decomposition is

$$\frac{x}{x^2+x-2} = \boxed{\frac{1}{3(x-1)}+\frac{2}{3(x+2)}}$$

d. First performing some polynomial division and then factoring the denominator, we get

$$\frac{x^6}{x^2-4} = x^4 + 4x^2 + 16 + \frac{64}{(x+2)(x-2)}$$

so we have reduced the problem to finding the partial fraction decomposition of the far right term. For this, we find the values of the unknown constants A and B in the equation

$$\begin{aligned}\frac{64}{(x+2)(x-2)} &= \frac{A}{x+2} + \frac{B}{x-2} \\ \Rightarrow \quad 64 &= A(x-2) + B(x+2)\end{aligned}$$

When $x = -2$ we get $A = -16$, and when $x = 2$ we get that $B = 16$. Thus the partial fraction decomposition is

$$\frac{x^6}{x^2-4} = \boxed{x^4 + 4x^2 + 16 - \frac{16}{x-2} + \frac{16}{x+2}}$$

e. Since the denominator is fully factored, we find the values of unknown constants A, B, C, D, E, and F in the following equation

$$\begin{aligned}\frac{x^4}{(x^2-x+1)(x^2+2)^2} &= \frac{Ax+B}{x^2-x+1} + \frac{Cx+D}{x^2+2} + \frac{Ex+F}{(x^2+2)^2} \\ \Rightarrow quad x^4 &= (Ax+b)(x^2+2)^2 + (Cx+D)(x^2-x+1)(x^2+2) \\ &\quad + (Ex+F)(x^2-x+1)\end{aligned}$$

Doing all sorts of horrendous distribution, we get

$$\begin{aligned}x^4 = {} & (A+C)x^5 + (B-C+D)x^4 + (4A+3C-D+E)x^3 + (4B-2C+3D-E+F)x^2 \\ & + (4A+2C-2D+E-F)x + (4B+2D+F)\end{aligned}$$

This yields the system of equations

$$\begin{aligned}A+C &= 0 \\ B-C+D &= 1 \\ 4A+3C-D+E &= 0 \\ 4B-2C+3D-E+F &= 0 \\ 4A+2C-2D+E-F &= 0 \\ 4B+2D+F &= 0\end{aligned}$$

and solving this with a matrix, we get

$$\text{rref}\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 1 \\ 4 & 0 & 3 & -1 & 1 & 0 & 0 \\ 0 & 4 & -2 & 3 & -1 & 1 & 0 \\ 4 & 0 & 2 & -2 & 1 & -1 & 0 \\ 0 & 4 & 0 & 2 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{4}{3} \\ 0 & 0 & 0 & 0 & 1 & 0 & \frac{4}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{4}{3} \end{pmatrix}$$

Thus $A = 0$, $B = -\frac{1}{3}$, $C = 0$, $D = \frac{4}{3}$, $E = \frac{4}{3}$, and $F = -\frac{4}{3}$, so the partial fraction decomposition is

$$\frac{x^4}{(x^2-x+1)(x^2+2)^2} = \boxed{-\frac{1}{3(x^2-x+1)} + \frac{4}{3(x^2+2)} + \frac{4x-4}{3(x^2+2)^2}}$$

5. After performing the partial fraction decomposition, we get

$$\begin{aligned}\int_0^1 \frac{2x+3}{(x+1)^2}\,dx &= \int_0^1 \frac{2}{x+1}\,dx + \int_0^1 \frac{1}{(x+1)^2}\,dx \\ &= \left[2\ln|x+1|\right]_0^1 + \left[-\frac{1}{x+1}\right]_0^1 \\ &= \boxed{2\ln(2) + \frac{1}{2} \approx 1.88629}\end{aligned}$$

6. Making the substitution

$$\begin{aligned}u &= e^x \\ du &= e^x\,dx\end{aligned}$$

we get

$$\begin{aligned}\int \frac{e^{2x}}{e^{2x}+3e^x+2}\,dx &= \int \frac{u}{u^2+3u+2}\,du \\ &= \int \frac{u}{(u+2)(u+1)}\,du\end{aligned}$$

For the partial fraction decomposition, we have to solve for constants A and B in the following equation

$$\begin{aligned}\frac{u}{(u+2)(u+1)} &= \frac{A}{u+2} + \frac{B}{u+1} \\ \Rightarrow \quad u &= A(u+1) + B(u+2).\end{aligned}$$

Setting $u = -2$ yields $A = 2$, and setting $u = -1$ yields $B = 1$. Putting this all together in the integral, we have

$$\begin{aligned}\int \frac{u}{(u+2)(u+1)}\,du &= \int \frac{2}{u+2} + \frac{1}{u+1}\,du \\ &= 2\ln|u+2| + \ln|u+1| + C \\ &= \boxed{2\ln|e^x+2| + \ln|e^x+1| + C}\end{aligned}$$