MAT265 Homework 10 (Solutions)

1. f is a polynomial, so it is continuous and differentiable on all of $\mathbb{R}$, and in particular, is continuous on $[-2, 2]$ and differentiable on $(-2, 2)$. Thus, f satisfies the hypotheses of the Mean Value Theorem (MVT). By MVT, there exists c in $(-2, 2)$ such that

$$\begin{aligned} f'(c) &= \frac{f(2) - f(-2)}{2 - (-2)} \\ 3c^2 - 3 &= \frac{[(2)^3 - 3(2) + 2] - [(-2)^3 - 3(-2) + 2]}{4} \\ 3c^2 - 3 &= \frac{4 - 0}{4} = 1 \\ c^2 &= \frac{4}{3} \\ c &= \pm\sqrt{\frac{4}{3}}. \end{aligned}$$

2. f is continuous on $[0, \infty)$ and differentiable on $(1, \infty)$, so in particular, it is continuous on $[1, 8]$ and differentiable on $(1, 8)$, thus satisfying the hypotheses for MVT. By MVT, there exists c in $(1, 8)$ so that

$$\begin{aligned} f'(c) &= \frac{f(8) - f(1)}{8 - 1} \\ \frac{2}{3}x^{-1/3} &= \frac{8^{2/3} - 1^{2/3}}{7} \\ \frac{2}{3}x^{-1/3} &= \frac{4 - 1}{7} = \frac{3}{7} \\ x^{-1/3} &= \frac{9}{14} \\ x &= \left(\frac{9}{14}\right)^{-3} = \frac{14^3}{9^3} = \frac{2744}{729} \approx 3.7641 \end{aligned}$$

3. Indeed, the average rate of change on $[0, 6]$ is 0 and there does not exist a c in the whole domain of f where $f'(c) = 0$. So there cannot exist a c for which

$$f'(c) = \frac{f(6) - f(0)}{6 - 0},$$

which rearranges to show that there cannot exist a c in the domain of f for which

$$f(6) - f(0) = f'(c)(6 - 0).$$

This does not contradict MVT because it does not apply: f is not differentiable at $x = 3$, so it does not satisfy the hypotheses of MVT on the interval $[0, 6]$.

4. We're given that f' exists for all real numbers, so in particular, f is continuous on $[0, 8]$ and differentiable on $(0, 8)$. By MVT we have that there exists a c in $[0, 8]$ for which

$$f'(c) = \frac{f(8) - f(0)}{8 - 0} = \frac{f(8) - f(0)}{8}.$$

Using this equality and the bounded property of f', we have

$$\begin{aligned} -1 &\le f'(x) \le 6 \\ -1 &\le f'(c) \le 6 \\ -1 \le \frac{f(8)-f(0)}{8} &\le 6 \\ -8 \le f(8)-f(0) &\le 48. \end{aligned}$$

5. **a.** Since the curve shown is the graph of f, we look for places where the concavity changes. This happens at $x \approx 3.7753$ and $x \approx 6.2247$.

b. Since the curve shown is the graph of f', we look for places where the graph changes from increasing to decreasing (and vice-versa). This happens at $x \approx 2.8787$, $x = 5$, and $x \approx 7.1213$.

c. Since the curve shown is the graph of f'', we look for places where the sign changes. This happens at $x = 2$ and $x = 8$.

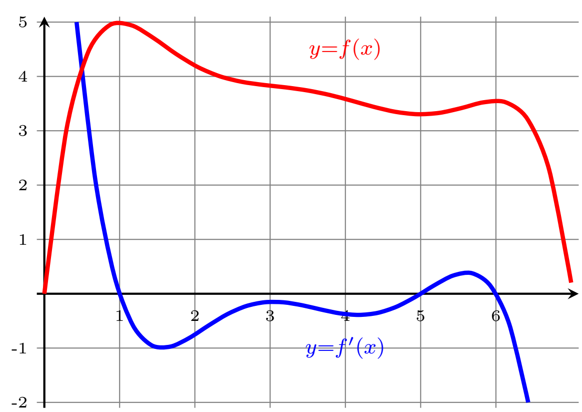
6. **a.** f is increasing on $(-\infty, 1)$ and $(5, 6)$; f is decreasing on $(1, 5)$ and $(6, \infty)$.

b. By the first derivative test, f has a local maximum at $x = 1$ and $x = 6$; f has a local minimum at $x = 5$.

c. The second derivative of f is the first derivative of this graph. So f is concave up on $(1.5677, 3.0455)$ and $(4.1736, 5.6133)$; f is concave down on $(-\infty, 1.5677)$, $(3.0455, 4.1736)$ and $(5.6133, \infty)$.

d. From the previous part, we see that f has inflection points at $x \approx 1.5677$, $x \approx 3.0455$, $x \approx 4.1736$, and $x \approx 5.6133$.

e. Assuming that $f(0) = 0$, the graph of f is below:



7. For $C(x) = x^{1/3}(x+4)$,

a. Taking the derivative, we have that

$$C'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3} = \frac{4x+4}{3x^{2/3}}.$$

Our critical points are thus $x = -1$ and $x = 0$. Testing values between these numbers, we have that $f'(x) < 0$ on $(-\infty, -1)$, $f'(x) > 0$ on $(-1, 0)$, and $f'(x) > 0$ on $(0, \infty)$. Thus f is increasing on $(-1, 0)$; f is decreasing on $(-\infty, -1)$ and $(0, \infty)$.

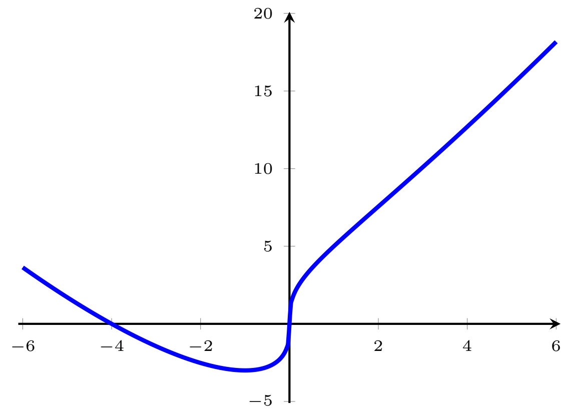
b. By the first derivative test in part (a), we have a local minimum at $x = -1$.

c. Taking the second derivative, we have

$$C''(x) = \frac{4(3x^{2/3}) - (4x+4)2x^{-1/3}}{9x^{4/3}} = \frac{4x-8}{9x^{5/3}}.$$

$C''(x) = 0$ when $x = 2$ and is undefined at 0. Testing points in between these numbers, we have that $f''(x) > 0$ on $(-\infty, 0)$, $f''(x) < 0$ on $(0, 2)$, and $f''(x) > 0$ on $(2, \infty)$. Thus f is concave upward on $(-\infty, 0)$ and $(2, \infty)$; f is concave downward on $(2, \infty)$. Thus $x = 0$ and $x = 2$ are inflection points for f.

d.



8. We have that f has critical points at $x = -7, -3, 4, 6$. Testing values in between these points, we get that $f'(x) > 0$ on $(-\infty, -7)$, $f'(x) > 0$ on $(-7, -3)$, $f'(x) < 0$ on $(-3, 4)$, $f'(x) > 0$ on $(4, 6)$, and $f'(x) > 0$ on $(6, \infty)$. So f is increasing on $(-\infty, 7)$, $(-7, -3)$, $(4, 6)$ and $(6, \infty)$.