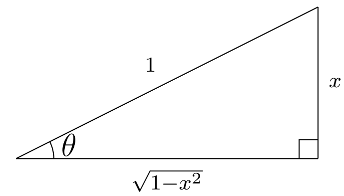
# MAT265 Homework 08 (Solutions)

**1.** Let $\theta = \sin^{-1} x$, so that $\sin\theta = x = \frac{x}{1}$. We then have the following right triangle:




From the picture we can see that

$$\tan\big(\sin^{-1}(x)\big) = \tan(\theta) = \frac{x}{\sqrt{1-x^2}}.$$

**2.** For simplicity, let $g(x) = x^2$, $h(x) = \ln(x)$, and $j(x) = \arcsin(x)$. We then have that

$$f(x) = g(x) \cdot h(j(x)),$$

and after applying a product rule and a chain rule, we get

$$\begin{aligned} f'(x) &= g'(x) \cdot h(j(x)) + g(x) \cdot h'(j(x)) \cdot j'(x) \\ &= 2x \ln(\arcsin(x)) + x^2 \cdot \frac{1}{\arcsin(x)} \cdot \frac{1}{\sqrt{1-x^2}}. \end{aligned}$$

**3.** **a.** First,

$$\lim_{x\to\infty} \frac{3+\sqrt{3}\,x^3}{4+2x^3} = \frac{\sqrt{3}}{2}.y$$

Since $\arccos(x)$ is continuous in a neighborhood of $\frac{\sqrt{3}}{2}$, we have

$$\lim_{x\to\infty} \arccos\left(\frac{3+\sqrt{3}\,x^3}{4+2x^3}\right) = \arccos\left(\lim_{x\to\infty}\frac{3+\sqrt{3}\,x^3}{4+2x^3}\right) = \arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}.$$

**b.** This limit is indeterminate of type $\frac{0}{0}$, so l'Hospital's Rule would apply. However, factoring is an easier method.

$$\lim_{x\to 2} \frac{x^3+x-10}{x-2} = \lim_{x\to 2}\frac{(x-2)(x^2+2x+5)}{x-2} = \lim_{x\to 2}(x^2+2x+5) = 13$$

**c.** This limit is indeterminate of type $\frac{0}{0}$, so l'Hospital's Rule would apply. However, using trig identities first might make it easier.

$$\lim_{x\to\frac{\pi}{2}} \frac{\cos(3x)}{\cot(5x)} = \lim_{x\to\frac{\pi}{2}} \frac{\cos(3x)\cdot\sin(5x)}{\cos(5x)}$$

This new limit is still indeterminate of type $\frac{0}{0}$, so applying l'Hospital's Rule, we get

$$\begin{aligned} \lim_{x\to\frac{\pi}{2}} \frac{\cos(3x)\cdot\sin(5x)}{\cos(5x)} &\overset{LH}{=} \lim_{x\to\frac{\pi}{2}} \frac{-3\sin(3x)\sin(5x)+5\cos(3x)\cos(5x)}{-5\sin(5x)} \\ &= \frac{-3(-1)(1)+5(0)(0)}{-5(1)} \\ &= -\frac{3}{5}. \end{aligned}$$

**d.** This limit is indeterminate of type $\frac{0}{0}$, so applying l'Hospital's Rule, we get

$$\lim_{x\to 0} \frac{x^3}{1-\cos(x)} \overset{LH}{=} \lim_{x\to 0} \frac{3x^2}{\sin(x)}.$$

This new limit is still indeterminate of type $\frac{0}{0}$, so applying l'Hospital's Rule again, we get

$$\lim_{x\to 0} \frac{3x^2}{\sin(x)} \overset{LH}{=} \lim_{x\to 0} \frac{6x}{\cos(x)} = \frac{6(0)}{1} = 0.$$

**e.** This limit is indeterminate of type $\frac{0}{0}$. Before applying l'Hospital's Rule, we make a substitution to save us some grief. Let $t = \sqrt{x}$. Then as $x \to \infty$, $t \to \infty$. So

$$\begin{aligned} \lim_{x\to\infty} \frac{\ln(\sqrt{x})}{\sqrt{x}} &= \lim_{t\to\infty} \frac{\ln(t)}{t} \\ &\overset{LH}{=} \lim_{t\to\infty} \frac{\left(\frac{1}{t}\right)}{1} = 0. \end{aligned}$$

**f.** This limit is super indeterminate, so we first rearrange it to get it into an indeterminate form that we know.

$$\lim_{t\to\infty} \frac{11^t - 5^t}{t} = \lim_{t\to\infty} \frac{5^t\left(\left(\frac{11}{5}\right)^t - 1\right)}{t}.$$

In this form, it is now clear that the limit is indeterminate of type $\frac{\infty}{\infty}$. So, applying l'Hospital's Rule, we get

$$\lim_{t\to\infty} \frac{5^t\left(\left(\frac{11}{5}\right)^t - 1\right)}{t}. \overset{LH}{=} \lim_{t\to\infty} \frac{5^t \ln 5\left(\left(\frac{11}{5}\right)^t - 1\right) + 5^t\left(\frac{11}{5}\right)^t \ln\frac{11}{5}}{1} = \infty.$$

**g.** This limit is indeterminate of type $0\cdot\infty$, and so l'Hospital's Rule does not directly apply. Rearranging the limit, we get

$$\lim_{x\to\infty} x^3 \tan\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\tan\left(\frac{1}{x}\right)}{1/x^3}.$$

This new limit is now indeterminate of type $\frac{0}{0}$, and so we can apply l'Hospital's Rule. Before applying it, we make a substitution to simplify the derivatives. Let $t = \frac{1}{x}$, so as $x \to \infty$, $t \to 0$.

$$\begin{aligned} \lim_{x\to\infty} \frac{\tan\left(\frac{1}{x}\right)}{1/x^3} &= \lim_{t\to 0} \frac{\tan t}{t^3} \\ &\overset{LH}{=} \lim_{t\to 0} \frac{\sec^2 t}{3t^2} = \infty. \end{aligned}$$

**h.** This limit is indeterminate of type $\infty^0$, so l'Hospital's does not apply directly. So, let

$$L = \lim_{x\to\infty} (e^x + 2x)^{\frac{1}{2x}}.$$

By continuity of the natural logarithm, we then have that

$$\ln L = \lim_{x\to\infty} \ln\Big((e^x + 2x)^{\frac{1}{2x}}\Big) = \lim_{x\to\infty} \frac{\ln(e^x + 2x)}{2x}.$$

This new limit is indeterminate of type $\frac{\infty}{\infty}$, so we can apply l'Hospital's Rule. Doing so, we get

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln(e^x + 2x)}{2x} &\overset{LH}{=} \lim_{x\to\infty} \frac{\left(\frac{e^x+2}{e^x+2x}\right)}{2} \\ &= \lim_{x\to\infty} \frac{e^x + 2}{2e^x + 4x}.\end{aligned}$$

Again this new limit is indeterminate of type $\frac{0}{0}$, so we apply l'Hospital's Rule again to get

$$\begin{aligned}\lim_{x\to\infty} \frac{e^x + 2}{2e^x + 4x} &\overset{LH}{=} \lim_{x\to\infty} \frac{e^x}{2e^x + 4} \\ &= \lim_{x\to\infty} \frac{e^x}{e^x\left(2 + \frac{4}{e^x}\right)} \\ &= \lim_{x\to\infty} \frac{1}{2 + \frac{4}{e^x}} = \frac{1}{2}.\end{aligned}$$

Thus we have $\ln L = \frac{1}{2}$, whence $L = e^{\frac{1}{2}}$.

**4.** This limit is indeterminate of type $\frac{0}{0}$, so applying l'Hospital's Rule, we get

$$\begin{aligned}\lim_{x\to\infty} \frac{5^x - 4^x}{3^x - 2^x} &\overset{LH}{=} \lim_{x\to\infty} \frac{5^x \ln 5 - 4^x \ln 4}{3^x \ln 3 - 2^x \ln 2} \\ &= \frac{\ln 5 - \ln 4}{\ln 3 - \ln 2}.\end{aligned}$$