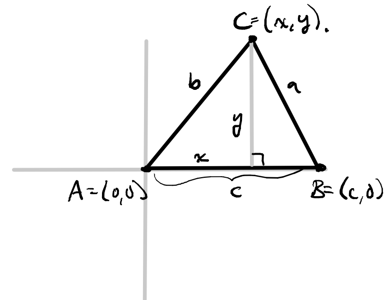
Section 6.2

Theorem (Law of Cosines) If ABC is a triangle with opposite sides of lengths a, b, c (resp.), then

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$



Proof Placing our triangle in the Cartesian plane (as pictured to the right), we have

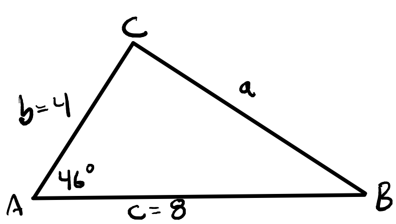
$$\cos A = \frac{x}{b} \Rightarrow x = b\cos A$$
$$\sin A = \frac{y}{b} \Rightarrow y = b\sin A.$$

By the distance formula, we have that

$$a = \sqrt{(x-c)^2 + (y-0)^2}$$
$$a^2 = (x-c)^2 + y^2$$
$$= x^2 - 2xc + c^2 + y^2$$
$$= b^2\cos^2 A - 2bc\cos A + c^2 + b^2\sin^2 A$$
$$= b^2(\cos^2 A + \sin^2 A) + c^2 - 2bc\cos A$$
$$= b^2 + c^2 - 2bc\cos A.$$

Playing this same game for the other two sides provides the result.

Ex

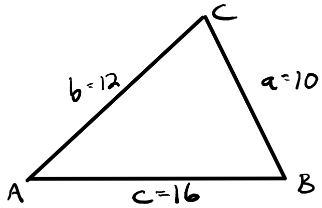


$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$a = \sqrt{16 + 64 - 64\cos(46^\circ)} = 5.96$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$\Rightarrow \cos B = \frac{a^2 + c^2 - b^2}{2ac} \approx 0.8759$$
$$\Rightarrow B \approx 28.85^\circ$$

$$C = 180^\circ - A - B = 180^\circ - 46^\circ - 28.85^\circ = 105.15^\circ$$

Ex



$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$\Rightarrow \cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{144+256-100}{2(12)(16)} = \frac{25}{32}$$
$$\Rightarrow \quad A \approx 38.62^\circ$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$\Rightarrow \cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{100+256-144}{2(10)(16)} = \frac{53}{80}$$
$$\Rightarrow \quad B \approx 48.51^\circ$$

$$C = 180^\circ - A - B = 180^\circ - 38.62^\circ - 48.51^\circ = 92.87^\circ$$

Theorem (Heron's Formula) Let a, b, c be the side lengths of a triangle. Then
$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)},$$
where $s = \frac{1}{2}(a+b+c)$.

Proof By the half-angle formula,
$$\cos\left(\frac{C}{2}\right) = \sqrt{\frac{1+\cos C}{2}} = \sqrt{\frac{1+\frac{a^2+b^2-c^2}{2ab}}{2}}$$

Law of Cosines, solve for $\cos C$

$$= \sqrt{\frac{2ab + a^2 + b^2 - c^2}{4ab}}$$
$$= \sqrt{\frac{(a+b)^2 - c^2}{4ab}}$$
$$= \sqrt{\frac{((a+b)+c)((a+b)-c)}{4ab}}$$

Now let $s = \frac{1}{2}(a+b+c)$, so that $a+b+c = 2s$. Similarly,
$$a+b-c = a+b+c-2c = 2s - 2c = 2(s-c),$$ so
$$= \sqrt{\frac{(2s)2(s-c)}{4ab}} = \sqrt{\frac{s(s-c)}{ab}}$$

BY A SIMILAR PROCESS, WE OBTAIN

$$\sin\left(\frac{C}{2}\right) = \sqrt{\frac{1-\cos C}{2}} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

RECALL NOW THAT $\text{AREA} = \frac{1}{2}ab\sin C$. So,

$$\text{AREA} = \frac{1}{2}ab\sin C = \frac{1}{2}ab \cdot 2\sin\left(\frac{C}{2}\right)\cos\left(\frac{C}{2}\right)$$

$$= ab\left(\sqrt{\frac{(s-a)(s-b)}{ab}}\right)\left(\sqrt{\frac{s(s-c)}{ab}}\right)$$

$$= ab\sqrt{\frac{s(s-a)(s-b)(s-c)}{(ab)^2}}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}$$

Ex FOR THE PREVIOUS TRIANGLE, $a=10$, $b=12$, $c=16$. So $s = \frac{1}{2}(10+12+16) = 19$, HENCE

$$\text{AREA} = \sqrt{19(19-10)(19-12)(19-16)} \approx 59.92 \text{ un}^2.$$