Section 2.6

Def Rational functions are quotients of polynomial functions $f(x) = p(x)/q(x)$. The domain is the set of all x-values for which the denominator is not zero.

Ex $f(x) = \dfrac{x^3 - 5x + 7}{(x-1)(x-3)}$ has domain $(-\infty, 1) \cup (1,3) \cup (3, \infty)$.
or $\{x \mid x \neq 1, 3\}$.

Arrow Notation

$x \to a^+$ means "x approaches a from the right."

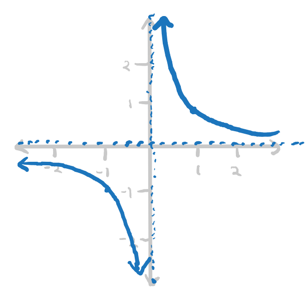
$x \to a^-$ means "x approaches a from the left."

$x \to \infty$ means "x increases without bound."

$x \to -\infty$ means "x decreases without bound."

This notation is useful because it allows us to talk about what happens near a point where a function is undefined.

Ex $\dfrac{1}{x^3}$ has domain $(-\infty, 0) \cup (0, \infty)$.
As $x \to 0^+$, $f(x) \to \infty$. We typically write $\lim\limits_{x \to 0^+} f(x) = \infty$. Similarly, $\lim\limits_{x \to 0^-} f(x) = -\infty$, and $\lim\limits_{x \to \infty} f(x) = 0$.

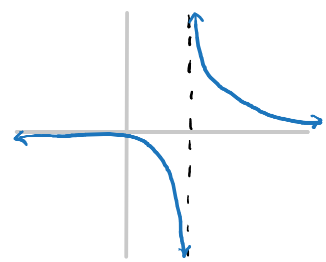


Def The vertical line $x = a$ is a vertical asymptote of f if $f(x)$ increases or decreases w/o bound as x approaches a.

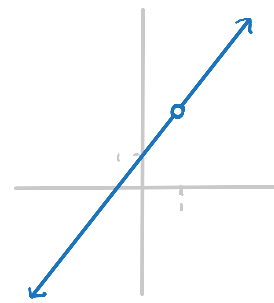
Note: If $x = a$ is a vertical asymptote of f, the graph of f <u>never crosses the line $x = a$</u>.

THEOREM Let $f(x) = p(x)/q(x)$ be a rational function. If $p(x)$ and $q(x)$ have no common factors and if a is a root of q, then $x=a$ is a vertical asymptote.

Ex $r(x) = \frac{1}{x-1}$. The numerator and denominator have no common factors, so $x=1$ is a vertical asymptote.



Ex $s(x) = \frac{x^2-1}{x-1}$. The numerator and denominator do have a common factor, and it is $x-1$, so $x=1$ is <u>not</u> a vertical asymptote. But the domain is $(-\infty,1)\cup(1,\infty)$ and on this domain $s(x) = \frac{x^2-1}{x-1} = x+1$, so $s(x)=x+1$ with a point missing at $x=1$.



DEF If $f(x) = \frac{p(x)}{q(x)}$ and $p(a)=q(a)=0$, $f(x)$ has a **hole** at $x=a$.

DEF A **horizontal asymptote** occurs when $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = c$ (or both), where b and c are (possibly the same) real number. The horizontal asymptotes are the lines $y=b$ and $y=c$.

NOTE: If $y=b$ is a horizontal asymptote of f, the graph of f <u>may cross the line $y=b$.</u>

Ex $\lim_{x\to\infty} \frac{x^2}{x} = \lim_{x\to\infty} \frac{x}{1} = \infty$

$\lim_{x\to-\infty} \frac{x^2}{x} = \lim_{x\to-\infty} \frac{x}{1} = -\infty$

$f(x) = \frac{x^2}{x}$ HAS NO HORIZONTAL ASYMPTOTES.

Ex $\lim_{x\to\infty} \frac{2x}{3x} = \lim_{x\to\infty} \frac{2}{3} = \frac{2}{3}$

$\lim_{x\to-\infty} \frac{2x}{3x} = \lim_{x\to\infty} \frac{2}{3} = \frac{2}{3}$

$g(x) = \frac{2x}{x}$ HAS A HORIZONTAL ASYMPTOTE OF $y = \frac{2}{3}$

Ex $\lim_{x\to\infty} \frac{x}{x^2} = \lim_{x\to\infty} \frac{1}{x} = 0$

$\lim_{x\to-\infty} \frac{x}{x^2} = \lim_{x\to-\infty} \frac{1}{x} = 0$

$h(x) = \frac{x}{x^2}$ HAS A HORIZONTAL ASYMPTOTE OF $y=0$.

Proposition LET $f(x) = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$ BE A RATIONAL FUNCTION.

1) IF $n > m$, THEN f HAS NO HORIZONTAL ASYMPTOTES.
2) IF $n = m$, THEN f HAS A HORIZONTAL ASYMPTOTE $y = \frac{a_n}{b_m}$.
3) IF $n < m$, THEN f HAS A HORIZONTAL ASYMPTOTE $y = 0$.

PROCEDURE TO PLOT RATIONAL FUNCTIONS $f(x) = \frac{p(x)}{q(x)}$

1. DETERMINE IF f HAS SYMMETRY (ie IS EVEN, ODD, NEITHER)
2. FIND ANY y-INTERCEPTS.
3. FIND ANY x-INTERCEPTS.
4. FIND ANY HOLES AND/OR VERTICAL ASYMPTOTES.
5. FIND ANY HORIZONTAL ASYMPTOTES.
6. PLOT AT LEAST ONE POINT TO THE LEFT AND RIGHT OF EACH ROOT AND VERTICAL ASYMPTOTE.