Section 2.1

Def The **imaginary unit** is defined to be $i = \sqrt{-1}$, where $i^2 = -1$.

Def A **complex number** is a number of the form $a + bi$, where a, b are real numbers and i is the imaginary unit. This is called the **standard form** for complex numbers, a is called the **real part**, and b is called the **imaginary part**.

Ex 1
$$\sqrt{-81} = \sqrt{(-1)81} = \sqrt{i^2 \cdot 9^2} = 9i$$
$$\sqrt{-3} = \sqrt{(-1)3} = \sqrt{i^2 \cdot 3} = i\sqrt{3},$$

Generally, to avoid confusion, we write i before a square root, so $3 + 6i\sqrt{2}$ is still in standard form.

Proposition Let $a+bi$, $c+di$ be complex numbers. Then $a+bi = c+di$ if and only if $a = c$ and $b = d$.

When adding/subtracting complex numbers, we treat the real and imaginary parts separately.
$$(a+bi) + (c+di) = (a+c) + (b+d)i$$
$$(a+bi) - (c+di) = (a-c) + (b-d)i$$

Ex 2
$$(11-2i) + (6+3i) = (11+6) + (-2+3)i = 17 + i$$
$$(11-2i) - (6+3i) = (11-6) + (-2-3)i = 5 - 5i$$

Since complex numbers are basically polynomials in i instead of x, they multiply in the same way:
$$(a+bi)(c+di) = ac + bci + adi + bdi^2 = (ac - bd) + (ad + bc)i,$$ since $i^2 = 1$.

Ex 3 $(3-4i)(5+7i) = 3\cdot 5 - 5\cdot 4i + 3\cdot 7i - 4\cdot 7i^2$

$= 15 - 20i + 21i + 28 = 43 + i$

DEF GIVEN A COMPLEX NUMBER $a+bi$, THE COMPLEX CONJUGATE OF $a+bi$ IS THE COMPLEX NUMBER $a-bi$. A HANDY FEATURE OF THE COMPLEX CONJUGATE IS THAT $(a+bi)(a-bi)$ IS A REAL NUMBER.

$(a+bi)(a-bi) = a^2 + abi - abi - b^2 i^2 = a^2 + b^2$.

DIVIDING BY COMPLEX NUMBERS REQUIRES THE USE OF THE COMPLEX CONJUGATE.

$$\frac{c+di}{a+bi} = \frac{c+di}{a+bi}\cdot\left(\frac{a-bi}{a-bi}\right) = \frac{(c+di)(a-bi)}{(a+bi)(a-bi)} = \frac{(c+di)(a-bi)}{a^2+b^2}$$

$$= \frac{(ac-bd)+(ad+bc)i}{a^2+b^2}$$

$$= \left(\frac{ac-bd}{a^2+b^2}\right) + \left(\frac{ad+bc}{a^2+b^2}\right)i$$

Ex 4 $\dfrac{2+3i}{5-2i} = \dfrac{(2+3i)(5+2i)}{(5+2i)(5-2i)} = \dfrac{10+15i+4i-6}{25+4} = \dfrac{4+19i}{29} = \dfrac{4}{29} + \dfrac{19}{29}i$

DEF GIVEN A REAL NUMBER $b>0$, THE PRINCIPAL SQUARE ROOT OF $-b$ IS DEFINED TO BE $\sqrt{-b} = i\sqrt{b}$

NOTE THE PRODUCT RULE FOR RADICALS ONLY APPLIES WHEN THE RADICANDS ARE <u>POSITIVE</u>. WHEN PERFORMING OPERATIONS INVOLVING SQUARE ROOTS OF NEGATIVE NUMBERS, <u>REWRITE THEM IN TERMS OF i FIRST</u>

THE REASONS FOR THIS ARE DEEPLY ROOTED IN COMPLEX ANALYSIS, SO FOR NOW IT WILL HAVE TO REMAIN SOMEWHAT MYSTERIOUS. BUT HERE IS AN EXAMPLE OF WHY IT MATTERS.

Ex 5 RIGHT: $\sqrt{-6} \cdot \sqrt{-6} = i\sqrt{6} \cdot i\sqrt{6} = i^2\sqrt{36} = -6$

WRONG: $\sqrt{-6} \cdot \sqrt{-6} = \sqrt{(-6)^2} = \sqrt{36} = 6$

AND CERTAINLY $6 \neq -6$!

THE REASON WE EVEN THOUGHT ABOUT COMPLEX NUMBERS IN THE FIRST PLACE IS BECAUSE WE WANTED TO FIND ROOTS OF THE POLYNOMIAL x^2+1. THIS LED TO US FINDING ROOTS OF MORE GENERAL QUADRATIC POLYNOMIALS.

RECALL: FOR THE EQUATION $ax^2+bx+c=0$, WITH $a \neq 0$, THE QUADRATIC FORMULA GIVES THE SOLUTIONS

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

Ex 6 THE SOLUTIONS $3x^2-5x+3=0$ ARE

$$\begin{aligned} x &= \frac{5 \pm \sqrt{25-4(3)(3)}}{2(3)} \\ &= \frac{5 \pm \sqrt{25-36}}{6} \\ &= \frac{5 \pm \sqrt{-11}}{6} \\ &= \frac{5}{6} \pm i\frac{\sqrt{11}}{6}. \end{aligned}$$