Nonarithmetic hybrids in PU(2, 1)

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Abstract

We explore hybrid subgroups of certain non-arithmetic lattices in PU(2, 1). In particular, we show that all of Mostow's small phase-shift non-arithmetic lattices arise via a hybrid construction

1 Introduction

One key notion in the study of lattices in a semisimple real Lie group G is that of *arithmeticity* (which we will not defined here; see [Mor15] for a standard reference). When G arises as the isometry group of a symmetric space X of non-compact type, the combined work of Margulis [Mar84], Gromov–Schoen [GS92], and Corlette [Cor92] imply that nonarithmetic lattices only exist when $X = \mathbf{H}^n_{\mathbb{R}}$ or $X = \mathbf{H}^n_{\mathbb{C}}$ (real and complex hyperbolic space, respectively); equivalently, up to finite index, $G = \mathrm{PO}(n, 1)$ or $\mathrm{PU}(n, 1)$. Due to their exceptional nature, it has been a major challenge to find and understand nonarithmetic lattices in these Lie groups.

Given two arithmetic lattices Γ_1, Γ_2 in $\mathrm{PO}(n, 1)$ with common sublattice $\Gamma_{1,2} \leq PO(n-1, 1)$, Gromov and Piatestki-Shapiro showed in [GP87] that one can produce a new "hybrid" lattice Γ in $\mathrm{PO}(n, 1)$ by way of a technique that they call "interbreeding." In particular, when Γ_1 and Γ_2 are not commensurable, Γ is shown to be nonarithmetic. It has been asked whether an analogous technique can exist for lattices in $\mathrm{PU}(n,1)$.

Hunt proposed one possible analog (see the references contained in [Pau12]) where one starts with two arithmetic lattices Γ_1, Γ_2 in $\mathrm{PU}(n,1)$ and embeddings $\iota_i : \mathrm{PU}(n,1) \hookrightarrow \mathrm{PU}(n+1,1)$ such that (1) $\iota_1(\Gamma_1)$ and $\iota_2(\Gamma_2)$ stabilize totally geodesic complex hypersurfaces in $\mathbf{H}^{n+1}_{\mathbb{C}}$, (2) that these hypersurfaces are orthogonal to one another, and (3) that $\iota_1(\Gamma_1) \cap \iota_2(\Gamma_2)$ is a lattice in $\mathrm{PU}(n-1,1)$. The hybrid subgroup is then $H(\Gamma_1, \Gamma_2) := \langle \iota_1(\Gamma_1), \iota_2(\Gamma_2) \rangle$.

In [Pau12] Paupert produces an infinite family of hybrids that are non-discrete. In [PW18] Paupert and the author produce examples of discrete (and arithmetic) hybrids in the Picard modular groups, some with finite covolume and others with infinite covolume. In this paper, we explore a slightly modified hybrid construction in the context of lattices produced by Mostow in [Mos80] (these are lattices in $\mathrm{PU}(2,1)$ generated by three complex reflections or order p with phase shift parameter t, see Section 2.3 for the explicit definition) and prove the following main result:

Theorem 1.

1. *All of Mostow's non-arithmetic lattices $\Gamma(p,t)$ with small phase shift ($|t| < \frac{1}{2} - \frac{1}{p}$) arise as hybrids of Fuchsian triangle groups.*
2. *The non-arithmetic lattices $\Gamma(4, 1/12)$ and $\Gamma(5, 1/5)$ in $\mathrm{PU}(2,1)$ arise as hybrids of two non-commensurable arithmetic lattices in $\mathrm{PU}(1,1)$.*

The second part of this theorem highlights the similarity of our hybrids and those hybrids of Gromov–Piatetski-Shapiro, specifically in its ability to produce a nonarithmetic lattice as a hybrid of two noncommensurable arithmetic lattices.

2 Background

2.1 Complex hyperbolic space

Let H be a Hermitian matrix of signature $(n,1)$ and let $\mathbb{C}^{n,1}$ denote $\mathbb{C}^{n+1}$ endowed with the Hermitian form $\langle\cdot,\cdot\rangle_H$ coming from H. Let V_- denote the set of points $z \in \mathbb{C}^{n,1}$ for which $\langle z,z\rangle_H < 0$, and let V_0 denote the set of points for which $\langle z,z\rangle_H = 0$. Given the usual projectivization map $\mathbb{P} : \mathbb{C}^{n,1} - \{0\} \to \mathbb{CP}^n$, *complex hyperbolic n-space* is $\mathbf{H}^n_{\mathbb{C}} = \mathbb{P}(V_-)$ with distance d coming from the Bergman metric

$$\cosh^2 \frac{1}{2} d(\pi(x), \pi(y)) = \frac{|\langle x,y\rangle|^2}{\langle x,x\rangle\langle y,y\rangle}$$

The ideal boundary $\partial_\infty \mathbf{H}^n_{\mathbb{C}}$ is then identified with $\mathbb{P}(V_0)$.

2.2 Complex hyperbolic isometries

Let $U(n,1)$ denote the group of unitary matrices preserving H. The holomorphic isometry group of $\mathbf{H}^n_{\mathbb{C}}$ is $\mathrm{PU}(n,1) = \mathrm{U}(n,1)/\,\mathrm{U}(1)$, and the full isometry group is generated by $\mathrm{PU}(n,1)$ and the antiholomorphic involution $z \mapsto \overline{z}$. Any holomorphic isometry of $\mathbf{H}^n_{\mathbb{C}}$ is one of the following three types:

- *elliptic* if it has a fixed point in $\mathbf{H}^n_{\mathbb{C}}$.
- *parabolic* if it has exactly one fixed point in the boundary (and no fixed points in $\mathbf{H}^n_{\mathbb{C}}$).
- *loxodromic* if it has exactly two fixed points in the boundary (and no fixed points in $\mathbf{H}^n_{\mathbb{C}}$).

Given a vector $v \in \mathbb{C}^{n,1}$ with $\langle v,v\rangle = 1$ and complex number ζ with unit modulus, the map

$$R_{v,\zeta}(x) : x \mapsto (\zeta - 1)\langle x,v\rangle v$$

is an an isometry of $\mathbf{H}^n_{\mathbb{C}}$ called a *complex reflection*, and its fixed point set $v^\perp \subset \mathbf{H}^n_{\mathbb{C}}$ is a totally geodesic subset called a $\mathbb{C}^{n-1}$*-plane* (or a *complex line* when $n = 2$).

2.3 Mostow's lattices

Following along with [Mos80] and [DFP05], $p \geq 3$ is an integer, t is a real number satisfying $|t| < 3\left(\frac{1}{2} - \frac{1}{p}\right)$, $\alpha = \frac{1}{2}\csc(\pi/p)$, $\varphi = e^{\pi i t/3}$, and $\eta = e^{\pi i/p}$. The Hermitian form is given by $\langle x, y\rangle = x^T H \overline{y}$ where

$$H = \begin{pmatrix} 1 & -\alpha\varphi & -\alpha\overline{\varphi} \\ -\alpha\overline{\varphi} & 1 & -\alpha\overline{\varphi} \\ -\alpha\varphi & -\alpha\overline{\varphi} & 1 \end{pmatrix}.$$

With p, t as above, the reflection groups to consider are $\Gamma(p,t) = \langle R_1, R_2, R_3\rangle$ where

$$R_1 = \begin{pmatrix} \eta^2 & -i\eta\overline{\varphi} & -i\eta\varphi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ -i\eta\varphi & \eta^2 & -i\eta\overline{\varphi} \\ 0 & 0 & 1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -i\eta\overline{\varphi} & -i\eta\varphi & \eta^2 \end{pmatrix}.$$

When $|t| < \frac{1}{2} - \frac{1}{p}$, Mostow refers to these groups has having "small phase shift."

Following the notation in [DFP05], we study closely related groups $\tilde{\Gamma}(p,t) = \langle R_1, J\rangle$ where

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

J has order 3 and $R_i = J R_{i+3} J^{-1}$ (where indices are taken modulo 3).

Proposition 2 (Lemma 16.1 in [Mos80], Prop 2.11 in [DFP05])**.** *If* $\Gamma(p,t)$ *is discrete, then it has index* 1 *or* 3 *in* $\tilde{\Gamma}(p,t)$.

Remark (Tables 1 and 2 in [Mos80], Remark 5.3 in [DFP05]). Given $p = 3, 4, 5$, there are only finitely-many values of t for which $\Gamma(p,t)$ is discrete. They are given in Table 1.

Theorem 3 (17.3 in [Mos80])**.** *The following lattices* $\Gamma(p,t) \leq \mathrm{PU}(2,1)$ *are nonarithmetic:* $\Gamma(3, 5/42)$*,* $\Gamma(3, 1/12)$*,* $\Gamma(3, 1/30)$*,* $\Gamma(4, 3/20)$*,* $\Gamma(4, 1/12)$*,* $\Gamma(5, 1/5)$*,* $\Gamma(5, 11/30)$*. The non-cocompact lattices* $\Gamma(p,t)$ *are arithmetic.*

p	$\lvert t\rvert < \frac{1}{2} - \frac{1}{p}$	$\lvert t\rvert \geq \frac{1}{2} - \frac{1}{p}$
3	$0, \frac{1}{30}, \frac{1}{18}, \frac{1}{12}, \frac{5}{42}$	$\frac{1}{6}, \frac{7}{30}, \frac{1}{3}$
4	$0, \frac{1}{12}, \frac{3}{20}$	$\frac{1}{4}, \frac{5}{12}$
5	$\frac{1}{10}, \frac{1}{5}$	$\frac{11}{30}, \frac{7}{10}$

Table 1: Values of p and t for which $\Gamma(p,t)$ is discrete.

3 Hybrids

We present a more general and flexible hybrid construction than that originally proposed by Hunt, with the hope that it allows us to produce more lattices.

Definition. Let $\Gamma_1, \Gamma_2 < \mathrm{PU}(n,1)$ be lattices. We define a *hybrid of* Γ_1, Γ_2 to be any group $H(\Gamma_1, \Gamma_2)$ generated by discrete subgroups $\Lambda_1, \Lambda_2 < \mathrm{PU}(n+1,1)$ stabilizing totally geodesic hypersurfaces Σ_1, Σ_2 (respectively) such that

1. Σ_1 and Σ_2 are orthogonal,
2. $\Gamma_i = \Lambda_i|_{\Sigma_i}$, and
3. $\Gamma_1 \cap \Gamma_2$ is a lattice in $\mathrm{PU}(n-1,1)$.

We note that the hybrids explored in [Pau12] and [PW18] are still hybrids in this new sense as well, taking $\Lambda_j = \iota_j(\Gamma_j)$ where ι_j was an embedding into Σ_j.

3.1 Small phase shift hybrids

When $\Gamma(p,t)$ has small phase shift, the fundamental domain for each of these groups is built by coning over a right-angled hexagon (see Figure 1 on Page 16 of [DFP05]) which becomes degenerate for larger t-values. Taking lifts to $\mathbb{C}^{2,1}$, one sees that non-adjacent sides for each hexagon intersect in positive vectors , which are given explicitly below:

$$v_{123} = \begin{pmatrix} -i\eta\overline{\varphi} \\ 1 \\ i\overline{\eta}\varphi \end{pmatrix}, \qquad v_{231} = \begin{pmatrix} i\overline{\eta}\varphi \\ -i\eta\overline{\varphi} \\ 1 \end{pmatrix}, \qquad v_{312} = \begin{pmatrix} 1 \\ i\overline{\eta}\varphi \\ -i\eta\overline{\varphi} \end{pmatrix},$$
$$v_{321} = \begin{pmatrix} i\overline{\eta\varphi} \\ 1 \\ -i\eta\varphi \end{pmatrix}, \qquad v_{132} = \begin{pmatrix} -i\eta\varphi \\ i\overline{\eta\varphi} \\ 1 \end{pmatrix}, \qquad v_{213} = \begin{pmatrix} 1 \\ -i\eta\varphi \\ i\overline{\eta\varphi} \end{pmatrix}.$$

Geometrically, v_{ijk} is the polar vector to the mirror for the complex reflection $J^{\pm 1}R_jR_k$ (for $k = j \pm 1$). What's more,

Proposition 4 (Proposition 2.13(3) in [DFP05])**.** *$v_{ijk} \perp v_{jik}$ and $v_{ijk} \perp v_{ikj}$.*

For the hybrid construction, we use the subspaces (considered as projective subspaces of $\mathbf{H}^2_{\mathbb{C}}$) corresponding to $v_{ijk}^{\perp}$. Since $Jv_{ijk} = v_{kij}$, it suffices to look only at $v_{312}^{\perp}$ and $v_{312}^{\perp}$ as the remaining subspaces are obtained by successive applications of J. In homogeneous coordinates, one sees that

$$v_{312}^{\perp} = \{[z, i\overline{\eta\varphi}, 1]^T \ : z \in \mathbb{C}\}$$
$$v_{321}^{\perp} = \{[i\overline{\eta}\varphi, z, 1]^T \ : \ z \in \mathbb{C}\}$$

Let $\Lambda_{ijk} \leq \Gamma(p,t)$ be the stabilizer subgroup of $v_{ijk}^{\perp}$, which is naturally identified with a subgroup of $\mathrm{PU}(1,1)$, and let Γ_{ijk} be a lift of this group to $\mathrm{SU}(1,1)$.

Proposition 5. *Γ_{312} is a cocompact lattice in $\mathrm{SU}(1,1)$. It is arithmetic for all pairs (p,t) appearing in Table 1 except $(3, 1/30)$, $(3, 1/12)$, $(3, 5/42)$, and $(4, 3/20)$.*

Proof. R_1 and R_3J both stabilize $v_{312}^{\perp}$:

$$R_1 : [z, i\overline{\eta\varphi}, 1]^T \mapsto [\eta^2 z + \overline{\varphi}^2 - i\eta\varphi, i\overline{\eta\varphi}, 1]^T$$
$$R_3J : [z, i\overline{\eta\varphi}, 1]^T \mapsto [i\overline{\eta\varphi}/z, i\overline{\eta\varphi}, 1]^T$$

Let A and B be the following elements in $\mathrm{SU}(1,1)$ corresponding to the actions of R_1 and R_3J on $v_{312}^{\perp}$, respectively.

$$A = \frac{1}{\eta}\begin{pmatrix} \eta^2 & \overline{\varphi}^2 - i\eta\varphi \\ 0 & 1 \end{pmatrix} \qquad B = \frac{1}{\sqrt{-i\overline{\eta\varphi}}}\begin{pmatrix} 0 & i\overline{\eta\varphi} \\ 1 & 0 \end{pmatrix}$$

One can check that

$$|\operatorname{Tr}(A)| = |e^{i\pi/p} + e^{-i\pi/p}|$$
$$|\operatorname{Tr}(B)| = 0$$
$$|\operatorname{Tr}(A^{-1}B)| = |e^{i\pi(-1/2+1/p+t/3)} + e^{-2\pi i t/3}|$$

All of these values are less than 2 when $p \geq 3$ and $|t| \neq \frac{1}{2} - \frac{1}{p}$ and so the elements are elliptic. Thus $\langle A, B\rangle$ is a cocompact triangle group (and therefore Γ_{312} is a cocompact lattice). By computing orders of these elements for (p,t) values in Table 1, one obtains table below showing the corresponding triangle groups, and arithmeticity of each can be checked by comparing with the main theorem of [Tak77].

(p,t)	$\triangle(x,y,z)$	(p,t)	$\triangle(x,y,z)$
$(3,-5/42)$	$\triangle(2,3,7)$	$(4,-3/20)$	$\triangle(2,4,5)$
$(3,-1/12)$	$\triangle(2,3,8)$	$(4,-1/12)$	$\triangle(2,4,6)$
$(3,-1/18)$	$\triangle(2,3,9)$	$(4,0)$	$\triangle(2,4,8)$
$(3,-1/30)$	$\triangle(2,3,10)$	$(4,1/12)$	$\triangle(2,4,12)$
$(3,0)$	$\triangle(2,3,12)$	$(4,3/20)$	$\triangle(2,4,20)$
$(3,1/30)$	$\triangle(2,3,15)$	$(5,-1/5)$	$\triangle(2,4,5)$
$(3,1/18)$	$\triangle(2,3,18)$	$(5,-1/10)$	$\triangle(2,5,5)$
$(3,1/12)$	$\triangle(2,3,24)$	$(5,1/10)$	$\triangle(2,5,10)$
$(3,5/42)$	$\triangle(2,3,42)$	$(5,1/5)$	$\triangle(2,5,20)$

□

Proposition 6. Γ_{321} *is a cocompact lattice in* $\mathrm{SU}(1,1)$. *It is arithmetic for all pairs* (p,t) *appearing in Table 1 except* $(3,-5/42)$, $(3,-1/12)$, $(3,-1/30)$, *and* $(4,-3/20)$.

Proof. R_2 and JR_3^{-1} both stabilize $v_{321}^{\perp}$:

$$R_2 : [i\overline{\eta}\varphi, z, 1]^T \mapsto [i\overline{\eta}\varphi, \eta^2 z + \varphi^2 - i\eta\overline{\varphi}, 1]^T$$
$$JR_3^{-1} : [i\overline{\eta}\varphi, z1]^T \mapsto [i\overline{\eta}\varphi, i\overline{\eta}\varphi/z, 1]^T$$

Let A and B be the following elements in SU(1,1) corresponding to the actions of R_2 and JR_3^{-1} on $v_{321}^{\perp}$, respectively.

$$A = \frac{1}{\eta}\begin{pmatrix} \eta^2 & \varphi^2 - i\eta\overline{\varphi} \\ 0 & 1 \end{pmatrix} \qquad B = \frac{1}{\sqrt{-i\overline{\eta}\varphi}}\begin{pmatrix} 0 & i\overline{\eta}\varphi \\ 1 & 0 \end{pmatrix}$$

One can check that

$$\begin{aligned} |\operatorname{Tr}(A)| &= |e^{i\pi/p} + e^{-i\pi/p}|, \\ |\operatorname{Tr}(B)| &= 0, \\ |\operatorname{Tr}(A^{-1}B)| &= |e^{i\pi(1/2+1/p-t/3)} - e^{2\pi i t/3}|, \end{aligned}$$

All of these values are less than 2 when $p \geq 3$ and $|t| \neq \frac{1}{2} - \frac{1}{p}$ and so the elements are elliptic. Thus $\langle A, B\rangle$ is a cocompact triangle group (and therefore Γ_{321} is a cocompact lattice). By computing orders of these elements for (p,t) values in Table 1, one obtains table below showing the corresponding triangle groups, and arithmeticity of each can be checked by comparing with the main theorem of [Tak77].

(p,t)	$\triangle(x,y,z)$	(p,t)	$\triangle(x,y,z)$
$(3,-5/42)$	$\triangle(2,3,42)$	$(4,-3/20)$	$\triangle(2,4,20)$
$(3,-1/12)$	$\triangle(2,3,24)$	$(4,-1/12)$	$\triangle(2,4,12)$
$(3,-1/18)$	$\triangle(2,3,18)$	$(4,0)$	$\triangle(2,4,8)$
$(3,-1/30)$	$\triangle(2,3,15)$	$(4,1/12)$	$\triangle(2,4,6)$
$(3,0)$	$\triangle(2,3,12)$	$(4,3/20)$	$\triangle(2,4,5)$
$(3,1/30)$	$\triangle(2,3,10)$	$(5,-1/5)$	$\triangle(2,5,20)$
$(3,1/18)$	$\triangle(2,3,9)$	$(5,-1/10)$	$\triangle(2,5,10)$
$(3,1/12)$	$\triangle(2,3,8)$	$(5,1/10)$	$\triangle(2,5,5)$
$(3,5/42)$	$\triangle(2,3,7)$	$(5,1/5)$	$\triangle(2,4,5)$

□

Theorem 7. *For $|t| < \frac{1}{2} - \frac{1}{p}$, the hybrid $H(\Gamma_{312}, \Gamma_{321})$ is the full lattice $\tilde{\Gamma}(p,t)$.*

Proof. The group $K = \langle R_1, R_3J, R_2, JR_3^{-1}\rangle$ is a subgroup of $H(\Gamma_{312}, \Gamma_{321})$. Since $J = (R_3J)^{-1}(JR_3^{-1})^{-1}$, $K = \langle R_1, J\rangle = \tilde{\Gamma}(p,t)$. □

By comparing with the table on Page 418 of [MR03], one sees that Γ_{312} and Γ_{321} are both arithmetic and noncommensurable in the cases where $(p,t) = (4, 1/12)$ and $(5, 1/5)$. This means that $\Gamma(4, 1/12)$ and $\Gamma(5, 1/5)$ are nonarithmetic lattices obtained by interbreeding two noncommensurable arithmetic lattices, exactly as in the Gromov–Piatetski-Shapiro construction.

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