Hyperbolic groups are finitely presented

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Abstract

In this short expository note we outline some of the basics geometric group theory and use them to prove that all hyperbolic groups have a finite presentation.

1 Groups into Metric Spaces

The central idea underlying geometric group theory is to use geometric tools to study groups. To do this, we must first construct some sufficiently nice geometric object that encodes the group. Throughout, we will assume that all groups are finitely-generated; this assumption is not always necessary, but it helps to avoid certain technicalities. For a more thorough introduction to geometric group theory and related metric spaces, see [1] or [2].

Definition. Let G be a group and with S the generating set. For any $g, h \in G$, define the *word metric* $d_S(g, h)$ to be the length of the shortest word in S representing $g^{-1}h$.

Proposition. *(G, d_S) is a metric space.*

Definition. Given a metric space (X, d) and an interval $[t_0, t_1] \subseteq \mathbb{R}$, a curve $\gamma : [t_0, t_1] \to X$ is a *geodesic* if $d(\gamma(t_0), \gamma(t_1)) = |t_0 - t_1|$ (where $|\cdot|$ is the usual absolute value on $\mathbb{R}$). We say X is a *geodesic metric space* (or a *length space*) if there exists a geodesic between every pair of points $x, y \in X$.

Definition. Let G be a group with generating set S. The *Cayley graph of G with respect to S*, $\Gamma_S(G)$, has vertex set G, and two vertices $g, h \in G$ are adjacent precisely when $g^{-1}h \in S$ or $h^{-1}g \in S$.

Although (G, d_S) is not itself a geodesic metric space, the Cayley graph $\Gamma_S(G)$ is, and the metric d_Γ is induced from d_S in the natural way: If g, h are adjacent

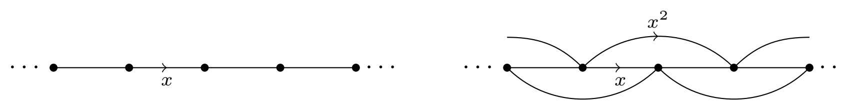

Figure 1: Cayley graphs for the cyclic group $\langle x \rangle$ with generating set $\{x\}$ (left) and $\{x, x^2\}$ (right)

vertices in $\Gamma_S(G)$, then $d_S(g,h) = 1$, so we declare that edges in the graph have length 1. Inductively, it follows that $d_S(g,h) = n$ precisely when the shortest path from g to h has length n, in which case these paths are geodesics in the Cayley graph, and we parameterize the edges accordingly.

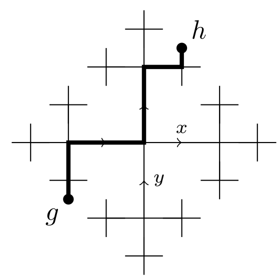

Figure 2: Geodesic path from g to h on the Cayley graph of the free group $F(\{x,y\})$

As one might expect from Figure 1, the metric space properties of the Cayley graph $\Gamma_S(G)$ rely on a specific choice of generating set. As it turns out, the effects of this choice are only discernible locally, and the large-scale behavior of this metric space is unaffected – we can overlook these choices by "squinting at the space" or "viewing it from a distance." We'll make this notion precise.

2 Quasi-Isometries

Definition. Let (X,d) and (X',d') be metric spaces, and let $\lambda \geq 1$, $\kappa \geq 0$. A map $f : X \to X'$ is called a (λ,κ)-*quasi-isometry* if both of the following hold:

(1) For every $x, y \in X$,

$$\frac{1}{\lambda} d(x,y) - \kappa \leq d'\left(f(x), f(y)\right) \leq \lambda d(x,y) + \kappa.$$

(2) There exists some $\varepsilon > 0$ such that for every $y \in X'$, there exists a corresponding $x \in X$ for which

$$d'(y, f(x)) \leq \varepsilon.$$

If such a quasi-isometry exists between (X,d) and (X',d'), we say that X and X' are *quasi-isometric*.

Proposition. *Quasi-isometry is an equivalence relation.*

The weakening of the isometry to the quasi-isometry allows us to regard non-isometric spaces as "the same(ish)". For example, $\mathbb{Z}$ and $\mathbb{R}$ with the usual absolute value are quasi-isometric as the canonical embedding $\mathbb{Z} \hookrightarrow \mathbb{R}$ is a $(1,0)$-quasi-isometry and the map $\mathbb{R} \to \mathbb{Z}$ given by rounding to the nearest integer is a $\left(1, \frac{1}{2}\right)$-quasi-isometry. It follows then that $\mathbb{Z}^n$ and $\mathbb{R}^n$, both with the usual metrics, are quasi-isometric.

Lemma 1. *For any group G with finite generating sets S and S', (G, d_S) and $(G, d_{S'})$ are quasi-isometric (thus $\Gamma_S(G)$ and $\Gamma_{S'}(G)$ are also quasi-isometric).*

Proof. Let λ be the maximum length of any generator $x \in S$ expressed as a word in S'. Then $\mathrm{Id} : G \to G$ is a $(\lambda, 0)$-quasi-isometry from (G, d_S) to (G, d_S) satisfying $d_{S'}(x, x) = 0$. $\square$

We may refer to two finitely generated groups as being quasi-isometric without ambiguity. It is natural to ask which properties, if any, may be quasi-isometry invariants. The title of this paper hints at one such property. The following definition was originally presented by Gromov in [4] (along with other equivalent characterizations).

Definition. We say that X is *δ-hyperbolic* (or just *hyperbolic*) if there exists $\delta \geq 0$ such that, for any triangle with edges that are geodesic (segments) γ_i $(i = 1, 2, 3)$ and for every $x \in \gamma_i$, then there exists $y \in \gamma_{i+1} \cup \gamma_{i+2}$ (with indices taken modulo 3) such that $x \in B_\delta(y)$. Such triangles are called *δ-slim.*

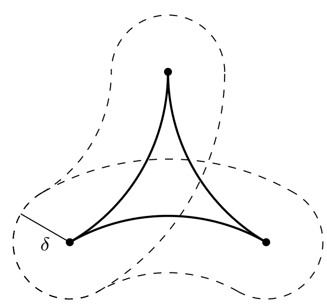

Figure 3: A δ-slim geodesic triangle

The definition of δ-hyperbolicity in a metric space is an attempt to coarsely model the behavior of negative curvature as in classical hyperbolic geometry. There are many other useful and interesting variants of this idea which are explored more thoroughly in [2].

Lemma 2. *Let (X, d) and (X', d') be quasi-isometric geodesic metric spaces. Then (X, d) is hyperbolic if and only if (X', d') is hyperbolic.*

The proof of this lemma is actually rather involved. In the interest of both brevity and clarity of exposition, we've chosen to just outline the argument; a complete argument can be found in [2, Ch. III.H.1]. The crucial objects in the proof are quasi-isometrically embedded real intervals.

Definition. Let (X, d) be a metric space. A curve $\gamma : [t_0, t_1] \to X$ is called a (λ, κ)-*quasi-geodesic* if for any subinterval $[\alpha, \beta]$ of $[t_0, t_1]$,

$$\frac{1}{\lambda}\ell\big(\gamma|_{[\alpha,\beta]}\big) - \kappa \leq d\left(\gamma(\alpha), \gamma(\beta)\right) \leq \lambda\ell\big(\gamma|_{[\alpha,\beta]}\big) + \kappa$$

where ℓ denotes the length of the curve.

Let $\gamma_1, \gamma_2, \gamma_3$ be geodesic segments (parameterized by arc length) in X which form a triangle Δ. As X is δ-hyperbolic for some δ, the triangle Δ is δ-slim. For a given (λ, κ)-quasi-isometry $f : X \to X'$, the curves $f(\gamma_1), f(\gamma_2), f(\gamma_3)$ are quasi-geodesics in X'. Since quasi-isometries do not distort distances too much, one can find a constant ε_1 (relying only on δ, λ, κ) for which $f(\Delta)$ is a ε_1-slim triangle in X'.

Now let $\gamma'_1, \gamma'_2, \gamma'_3$ be geodesic segments in X' with the same endpoints as $f(\gamma_1), f(\gamma_2), f(\gamma_3)$ (respectively). One can also find another constant ε_2 (relying only on δ, λ, κ) so that each $f(\gamma_i)$ is contained in a neighborhood of radius ε_2 around γ'_i.

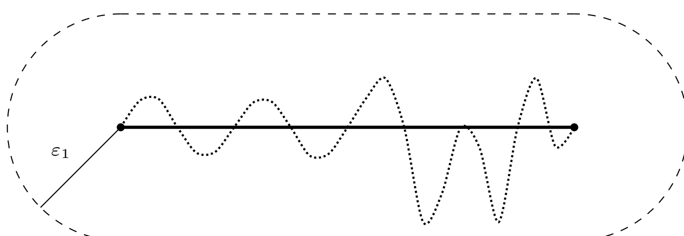

Figure 4: Quasi-geodesic segments (dotted) are uniformly close to geodesic segments (solid) with the same endpoints

In this way, with $\Delta' \subset X'$ the triangle formed by these geodesics γ'_i and with $\delta' = \varepsilon_1 + \varepsilon_2$, we have that Δ' is a δ'-slim triangle. Since δ' relies only on δ, λ, κ, it follows that X' is δ'-hyperbolic.

3 Hyperbolic Groups

Definition. For a group G with generating set S, if the Cayley graph $\Gamma_S(G)$ is a hyperbolic geodesic metric space, then G is called a *hyperbolic group*.

Hyperbolic groups do arise rather naturally. For example, every finite group is hyperbolic as the Cayley graphs are bounded. Free groups are hyperbolic as the Cayley graphs are trees (see Figure 2 for example), and subgroups of free groups are hyperbolic since they are also free. More generally, finite index subgroups of hyperbolic groups are themselves hyperbolic, but as Rips showed in 1982, the same cannot be said about arbitrary subgroups (see [5]).

Before stating the main theorem of this note, which is also attributed to Rips (see [3, Ch. 5,Thm. 2.3]), we first recall the notion of a group presentation.

Definition. Let $F(S)$ be the free group with generating set S, and let R be a (possibly infinite) set of words in $F(S)$. The group G defined as the quotient of $F(S)$ by the normal subgroup generated by R is denoted $G = \langle S|R\rangle$, and we call this notation a *presentation for* G. Elements of S are called *generators* for G and elements of R are called *relators*. $G = \langle S \mid R\rangle$ is *finitely presented* if both S and R are finite sets.

Remark. Presentations are not unique, and it is a notoriously difficult problem in general to determine if two presentations define isomorphic groups. Hyperbolic groups comprise one such family of groups where this problem is actually solvable (see [2, Ch. III.Γ.2]).

Theorem. *Every hyperbolic group is finitely presented.*

Proof. Let $\delta > 0$, let G be a δ-hyperbolic group, and fix some finite generating set S. Let d_S be the word metric on G. For each $k \in \mathbb{Z}^+$, define

$$\begin{aligned}
B_k &:= \{g \in G \,:\, d_S(g, 1) \leq k\}\\
R_k &:= \{xyz \,:\, x, y, z \in B_k, xyz = 1 \in G\} \cup \{xx^{-1} \,:\, x \in B_k\}\\
G_k &:= \langle B_k \mid R_k\rangle.
\end{aligned}$$

Since $B_1 \subseteq B_2 \subseteq \cdots$ and $R_1 \subseteq R_2 \subseteq \cdots$, we obtain the following sequence of group homomorphisms

$$G_1 \xrightarrow{\varphi_1} G_2 \xrightarrow{\varphi_2} \cdots \longrightarrow G_\infty = G$$

Our goal is to show that φ_N is an isomorphism for sufficiently large N, from which it will follow that $G = \langle B_N \mid R_N\rangle$. First, we show that each φ_k is surjective. Choose $g \in B_{k+1} - B_k$, so $g = s_1 \cdots s_{k+1}$ for $s_i \in S$. Then there exist $x, y \in B_k$ (say $x = s_{k+1}^{-1}$ and $y = s_k^{-1} \cdots s_1^{-1}$) such that $xyg = 1 \in G$. Since $x, y, g \in B_{k+1}$, we have that $xyg \in R_{k+1}$.

To see injectivity, fix $N \gg 2\delta$ and suppose that $xyz \in R_{N+1}$. We aim to show that this relation can be deduced from R_N, but x, y, z need not be in B_N. To get around this, we may choose $x_1, x_2 \in B_N$ such that $x = x_1x_2$ (called a *splitting* of

x) and such that $d_S(x_1,1) > \delta$, $d_S(x_2,1) > \delta$, and $d_S(x,1) = d_S(x_1,1)+d_S(x_2,1)$ (called the *canonical splitting* of x). Then, adding the generator x and the relation $x_1x_2x^{-1}$ to the presentation for G_N, we get an equivalent presentation. From here, we now show how to deduce $xyz = 1$ from the relations in B_N.

(Case 1.) Suppose $x, y \in B_N$ and $z \in B_{N+1} - B_N$. Choose a canonical splitting $z = z_1z_2$ and let P be the point corresponding to the choice of z_1 and z_2. Since the geodesic triangle Δ (in the Cayley graph $\Gamma_S(G)$) with vertices $1, x, xy$ is δ-thin, there exists a point Q on one of the other edges that is within δ of P - without loss of generality, suppose $Q \in [1, x]$. Then the geodesic $[P, Q]$ and the geodesic $[Q, xy]$ divide Δ into three smaller triangles, all with sides of length at most N. It follows then that the relation $xyz_1z_2 = 1$ can be deduced from three relations in R_N.

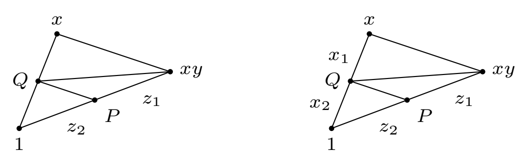

Figure 5: Partitioning Δ in Case 1 and Case 2, respectively

(Case 2.) Suppose $y, z \in B_{N+1} - B_N$. By Case 1, assume all true relations of the form $abc = 1$ for $a, b \in X_N$ and $c \in X_{N+1}$. Once again, choose a canonical splitting $z = z_1z_2$ as above, and proceed similarly. Here, if Q lies on an edge of length $N+1$, then it corresponds to some splitting of either x or y - suppose $x = x_1x_2$ (which we can assume as both x_1 and x_2 have lengths at most N). Once again, we divide the triangle Δ into three smaller triangles. With this division, it's possible that one of the triangles has a single side of length $N + 1$. However, all of the other geodesic segments have length at most N, so by Case 1, we are done

Similar arguments apply to the relations of the form $xx^{-1}, x \in B_{N+1} - B_N$, thus completing the proof. $\square$

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